On Palindromic Length of Sturmian Sequences

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Palindromic length

Let $w \in A^*$.

$$|w|_{\text{pal}} = \min \left\{ k \in \mathbb{N} \ : \ w = p_1 p_2 \cdots p_k, \text{ every } p_i \text{ is a palindrome} \right\}$$

Example. $w = baabab$

Let $\mathcal{L} \subset A^*$.

$$\text{pal}_{\mathcal{L}}(n) = \max \left\{ |w|_{pal} \ : \ w \in \mathcal{L} \text{ and } |w| = n \right\}$$

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Conjecture (Frid, Puzynina, Zamboni, 2013)

Let $\mathcal{L}$ be the language of an infinite word $\boldsymbol{u}$.

$$\mathrm{pal}_{\mathcal{L}}(n) \text{ is bounded} \quad \Longrightarrow \quad \boldsymbol{u} \text{ is eventually periodic}$$

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Main results

Let $\mathcal{L}$ be the language of a Sturmian word. Then $\mathrm{pal}_{\mathcal{L}}$ may grow into infinity arbitrarily slow...

Theorem

Let $f : \mathbb{N} \to \mathbb{R}$ be a non-decreasing function with $\lim_{n\to\infty} f(n) = +\infty$. Then there is a Sturmian word $\boldsymbol{u}$ such that $\mathrm{pal}_{\mathcal{L}(u)}(n) = \mathcal{O}(f(n))$.

...and not faster than $\mathcal{O}(\ln n)$.

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There is a constant K such that for every Sturmian word $\boldsymbol{u}$ we have $\mathrm{pal}_{\mathcal{L}(u)}(n) \leq K \ln n$.

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Sturmian words

- binary infinite words that have exactly $n+1$ factors of length n for each $n \geq 0$

- the slope of a Sturmian word $\boldsymbol{u} = (u_n)_{n \geq 0}$

$$\rho(\boldsymbol{u}) = \lim_{n \to \infty} \pi(u_0 \cdots u_{n-1}) = \frac{|u_0 \cdots u_{n-1}|_1}{n}$$

is well defined and is irrational

- $\boldsymbol{u}, \boldsymbol{v}$ Sturmian words

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Sturmian morphisms

The set of morphisms preserving Sturmian words is the monoid generated by

$$E : \begin{matrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{matrix} \qquad G : \begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 01 \end{matrix} \qquad \widetilde{G} : \begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 10 \end{matrix}$$

Definition

Let $\psi_b : \{0,1\}^* \to \{0,1\}^*$, where $b \in \mathbb{N}$, $b \geq 1$ and

$$\psi_b(0) = 10^{b-1},$$
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Remark. $\psi_b = \widetilde{G}^{b-1} \circ E \circ G$.

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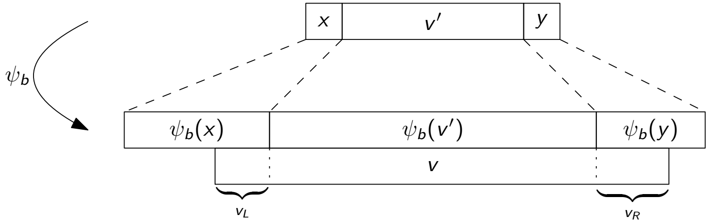
Images of Sturmian words

Lemma

Sturmian word $\boldsymbol{u}$, $\rho(\boldsymbol{u}) = [0, b, a_1, a_2, a_3, \ldots]$. A factor $v \in \mathcal{L}(\boldsymbol{u})$, $|v|_1 \geq 2$.

Then $\exists v', v_L, v_R$: $v' \neq \varepsilon$ is a factor of a Sturmian word with slope $\alpha = [0, a_1, a_2, a_3, \ldots]$, v_L is a proper suffix of $\psi_b(x)$ and v_R is a proper prefix of $\psi_b(y)$ for some $x, y \in \{0, 1\}$, and

1. $v = v_L \psi_b(v') v_R$,
2. $|v|_{\text{pal}} \leq 4 + |v'|_{\text{pal}}$.



Idea of proofs

Sturmian word $\boldsymbol{u}$ with the slope $\rho = [0, a_1, a_2, \ldots]$, $v \in \mathcal{L}(\boldsymbol{u})$.

Successive application of Lemma: words $v = v^{(1)}, v^{(2)}, \ldots, v^{(j+1)}$ such that

1. $v^{(i)}$ is a factor of a St. word with slope $[0, a_i, a_{i+1}, \ldots]$,
2. $|v^{(i)}| \geq |\psi_{a_i}(v^{(i+1)})| \geq a_i |v^{(i+1)}|$,
3. $|v^{(i)}|_{\text{pal}} \leq 4 + |v^{(i+1)}|_{\text{pal}}$,
4. $v^{(j+1)}$ does not contain two ones.

Altogether we have

$$|v| = |v^{(1)}| \geq a_1 a_2 \cdots a_j,$$
$$|v|_{\text{pal}} \leq 4j + 2.$$

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Arbitrarily slow growth

$f : \mathbb{N} \to \mathbb{R}$ be a non-decreasing function with $\lim_{n\to\infty} f(n) = +\infty$.

Find $(a_i)_{i\geq 1}$ such that

$$f(a_1) \geq 1$$
$$f(a_1 a_2) \geq 2^2$$
$$\vdots$$
$$f(a_1 a_2 \cdots a_k) \geq k^2$$

Then

$$\frac{|v|_{\text{pal}}}{f(|v|)} \leq \frac{4j+2}{f(a_1 a_2 \cdots a_j)} \leq \frac{4j+2}{j^2}.$$

Therefore

$$\limsup_{n\to\infty} \frac{\text{pal}_{\mathcal{L}(u)}(n)}{f(n)} \leq \lim_{j\to\infty} \frac{4j+2}{j^2} = 0.$$

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Growth not faster than $\mathcal{O}(\ln n)$

Word $v^{(i)}$ contains factor $(\psi_{a_i} \circ \psi_{a_{i+1}})(v^{(i+2)})$.

$|v^{(i)}| \geq 2|v^{(i+2)}|$ and thus $|v| \geq 2^{\lfloor \frac{j}{2} \rfloor}$.

Using this estimate we get

$$\frac{|v|_{\mathrm{pal}}}{\ln|v|} \leq \frac{4j+2}{\frac{j-1}{2}\ln 2} \xrightarrow{j\to\infty} \frac{8}{\ln 2}.$$

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Palindromic length and class $\mathcal{P}$

A primitive morphism $\varphi : A^* \mapsto A^*$ is in class $\mathcal{P}$,
if there is a palindrome p such that for each $a \in A$

$$\varphi(a) = pq_a, \text{ where } q_a \text{ is a palindrome.}$$

Example

$$\varphi_F = \begin{cases} a \mapsto ab \\ b \mapsto a \end{cases} \quad \varphi_{TM} = \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases} \quad \varphi_{TM}^2 = \begin{cases} a \mapsto abba \\ b \mapsto baab \end{cases}$$

Remark. A fixed point of a primitive morphism fulfills assumptions of Frid-Puzynina-Zamboni, thus $\mathrm{pal}_{\mathcal{L}}(n)$ is unbounded.

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Theorem [A., Kadlec, Masáková, Pelantová (2019)]

Let $\psi : A^* \to A^*$, $\psi \in \mathcal{P}$ s.t. for each $a \in A$ $\psi(a) = q_a p$, where $p \in \{\varepsilon\} \cup A$ and q_a is a palindrome. Let us denote

$$C := \max\{|x|_{\text{pal}} : \exists\, a \in A, x \text{ is a proper prefix of } q_a\}.$$

Then for the language $\mathcal{L}$ of a fixed point of ψ we have

$$\limsup_{n\to\infty} \frac{\text{pal}_{\mathcal{L}}(n)}{\ln n} \leq \frac{2C + \frac{3}{2}|p|}{\ln \Lambda},$$

where Λ is the dominant eigenvalue of the incidence matrix of ψ.

Remark. By a result of Allouche, $|p| \leq 1$ is not a restriction.

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Fibonacci word

- Fibonacci word $\boldsymbol{f}$, fixed point of $\varphi_F : a \mapsto ab,\ b \mapsto a$
- Frid: $\limsup_{n\to\infty} \mathrm{pal}_{\mathcal{L}(\boldsymbol{f})}(n) = +\infty$
- Application of Theorem

$$\limsup_{n\to\infty} \frac{\mathrm{pal}_{\mathcal{L}(\boldsymbol{f})}}{\ln n} \leq \frac{3}{2\ln\tau} \qquad (\text{for } \varphi_F)$$

$$\leq \frac{4}{3\ln\tau} \qquad (\text{for } \psi = \varphi_F^3)$$

$$\leq \frac{1}{\ln\tau} \qquad (\text{for } \psi^2)$$

$$\leq \frac{2(k+1)}{3k\ln\tau} \qquad (\text{for } \psi^k, k = 1, \ldots, 13)$$

- Conjecture

$$\limsup_{n\to\infty} \frac{\mathrm{pal}_{\mathcal{L}(\boldsymbol{f})(n)}}{\ln n} = \frac{2}{3\ln\tau}$$

Fibonacci word

- Fibonacci word $\boldsymbol{f}$, fixed point of $\varphi_F : a \mapsto ab,\ b \mapsto a$
- Frid: $\limsup_{n\to\infty} \mathrm{pal}_{\mathcal{L}(\boldsymbol{f})}(n) = +\infty$
- Application of Theorem

$$\begin{aligned}\limsup_{n\to\infty} \frac{\mathrm{pal}_{\mathcal{L}(\boldsymbol{f})}}{\ln n} &\leq \frac{3}{2\ln\tau} && (\text{for } \varphi_F)\\ &\leq \frac{4}{3\ln\tau} && (\text{for } \psi = \varphi_F^3)\\ &\leq \frac{1}{\ln\tau} && (\text{for } \psi^2)\\ &\leq \frac{2(k+1)}{3k\ln\tau} && (\text{for } \psi^k, k = 1,\ldots,13)\end{aligned}$$

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Thue-Morse word

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$$\limsup_{n\to\infty} \frac{\mathrm{pal}_{\mathcal{L}(\boldsymbol{t})}(n)}{\ln n} \le \frac{4}{\ln 4} \qquad (\text{for } \varphi_{TM})$$

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[proved to be true by A. Frid, DLT 2019, LNCS 11647, 234–243]

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Thank you for your attention.