Uniwersytet Wrocławski

Deciding Context Unification with regular constraints

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Developments in Language Theory
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Equations over terms

- Equations over terms (unification)

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- What can the variables represent?

(First order) term unification

Variables represent closed terms

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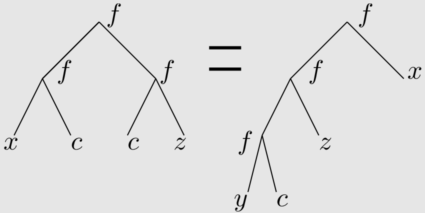

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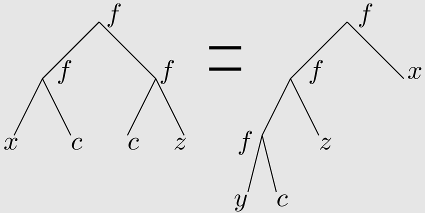
(First order) term unification

Variables represent closed terms: polynomial



- iterative decomposition

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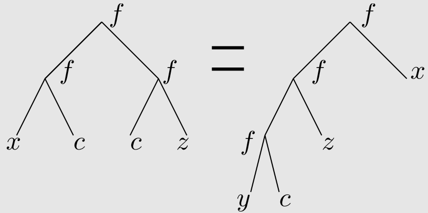

- iterative decomposition

Not covered: “functions”

- open terms
- λ-terms
- ...

Second order unification

- Variables have arguments that can be used. Terms with ‘holes’.

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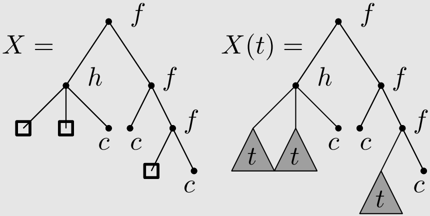
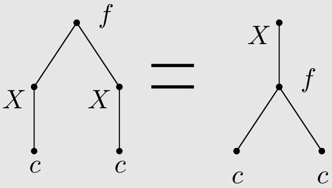


Example

Equation



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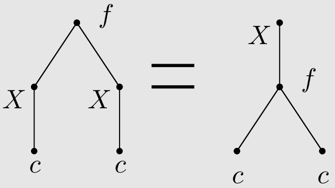
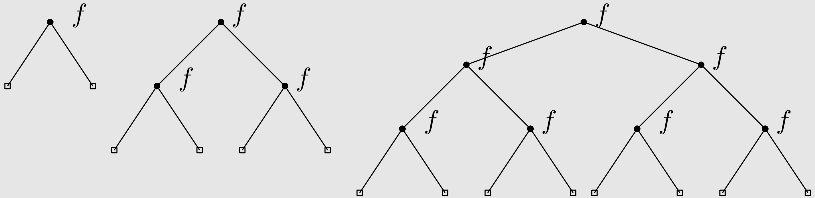


Solutions: full binary trees



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Undecidable.

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- one argument
- one binary symbol
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Subtlety

- unbounded number of usages (this causes hardness)
- can the argument be ignored (easy)

Word equations

- Unary signature: word equations.
- Variables represent words $\in \Sigma^*$.
- $aXbXYbbb = XabaabYbY$ $(X = aa, Y = bb)$

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Restrictions of second order unification

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Decidable. [Makanin $\to$ Plandowski (PSPACE)]

Context unification

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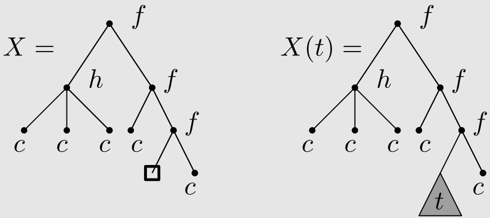
Context unification

Second order unification

Substitution for X uses argument exactly once.



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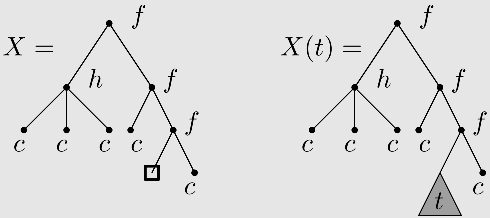

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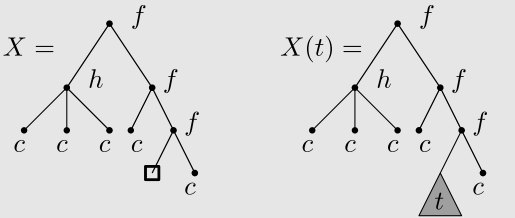

Easy reduction: k-arguments $\to$ one argument (NP-reduction)

Regular constraints: only substitution from a regular (tree) language.

More formally

Definition (Context unification)

- signature (fixed arities) $[f/2, a/1, c/0]$
- context variables denoting terms with one 'hole' $[X/1]$
- term variables denoting closed terms $[x/0]$
- equations built with them

More formally

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Definition (Substitution)

A substitution S assigns

- to a term variable: a closed term over a signature
- to a context variable: a term over a signature plus a unique occurrence of 'hole' $(\bullet)$
- extend in a natural way
  $S(X(t))$: in $S(X)$ we replace $\bullet$ with $S(t)$.

Example

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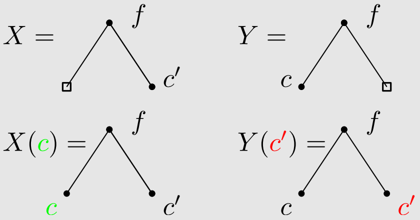

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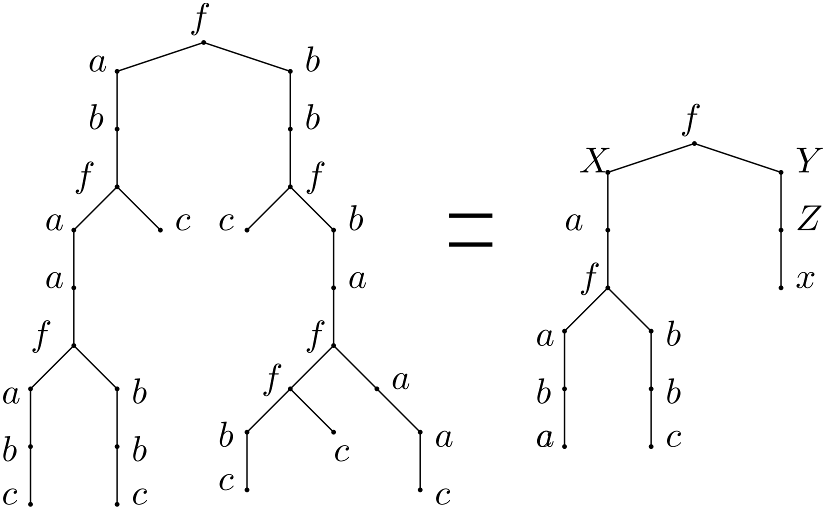

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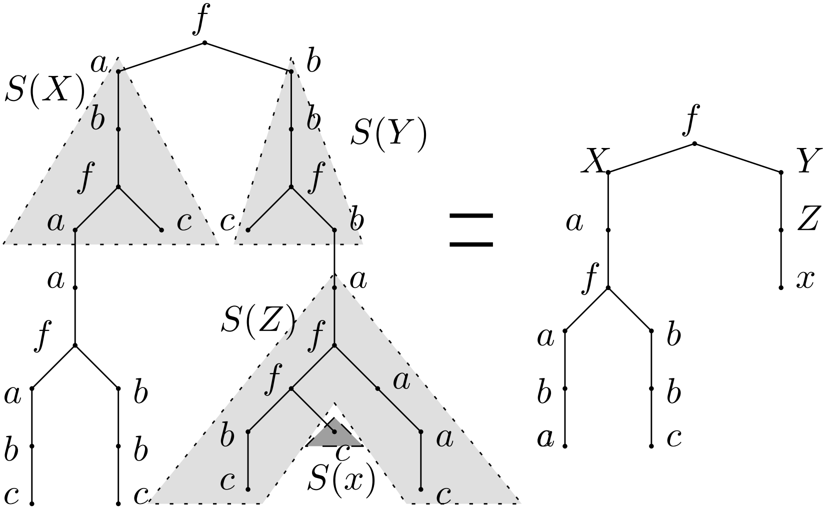

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Why regular constraints?

- because we can
- more expressive ones $\implies$ undecidability
- equivalent to linear second order unification
- regular constraints are somehow expressive
- very successful for word equations

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- representation of Ls matters for computational complexity.
- some representations are easier to handle

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In between important problems:

- word equations , first order-unification
- second-order unification

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- Unique with this property
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Connections

- one-step term rewriting
- natural language parsing
- linear second-order unification =
  context unification + regular constraints

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- special cases
  - one context variable
  - two context variables
  - stratified context unification
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We will not use any of that.

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With regular constraints: in EXPTIME
Easily EXPTIME-hard.

Preliminary notes: signature

- The solution may have elements not present in the equation:
  - $\Sigma = \{f/2, c/0, c'/0\}$
  - $X(c) = Y(c')$
  - $S(X) = f(\bullet, c')$, $S(Y) = f(c, \bullet)$

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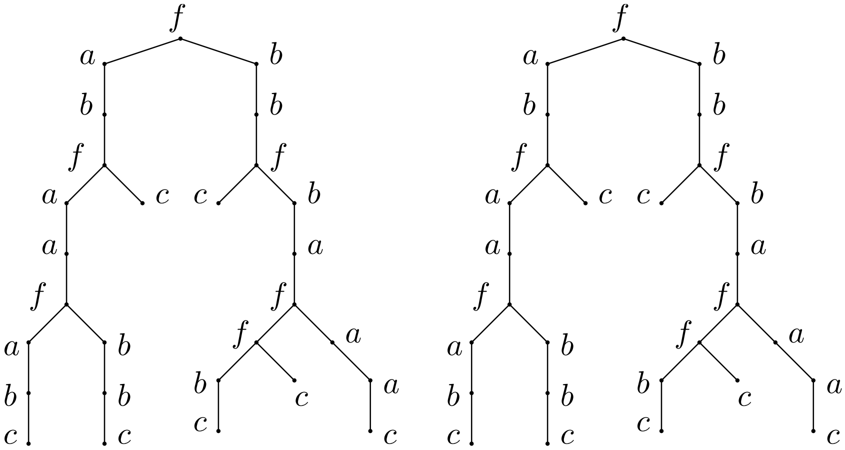
- the solution uses other letter f/k: replace it with a fixed f_k/k
- when f_k/k is not in the equation:
  - replace $f_k(t_1, \dots, t_k)$ with $g(t_1, g(\dots g(t_{k-1}, t_k)\dots))$
  - $f_1(t_1) \to t_1$
  - $c_0 \to c$

Local compressions



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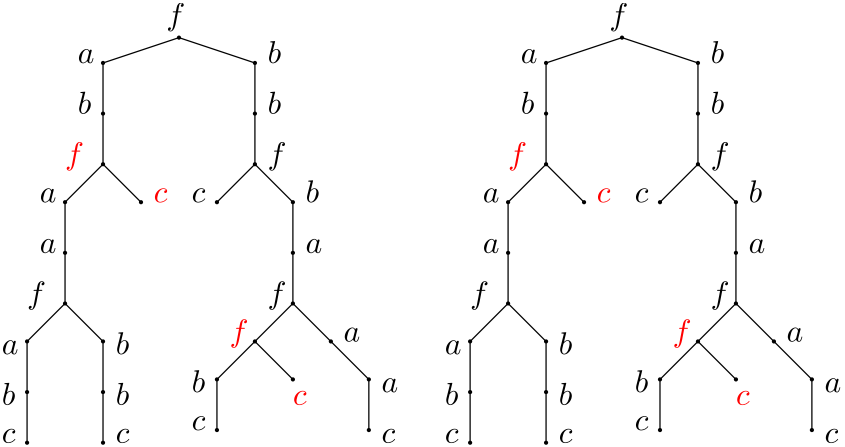

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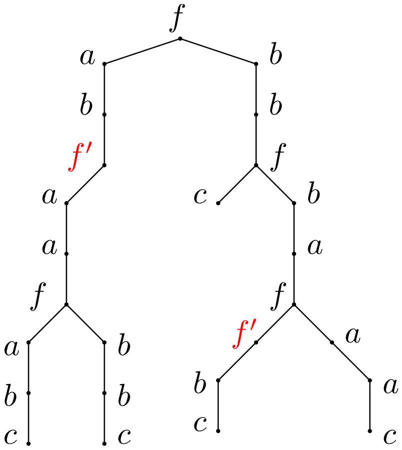

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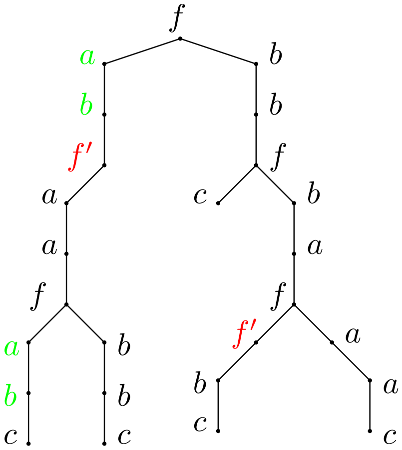

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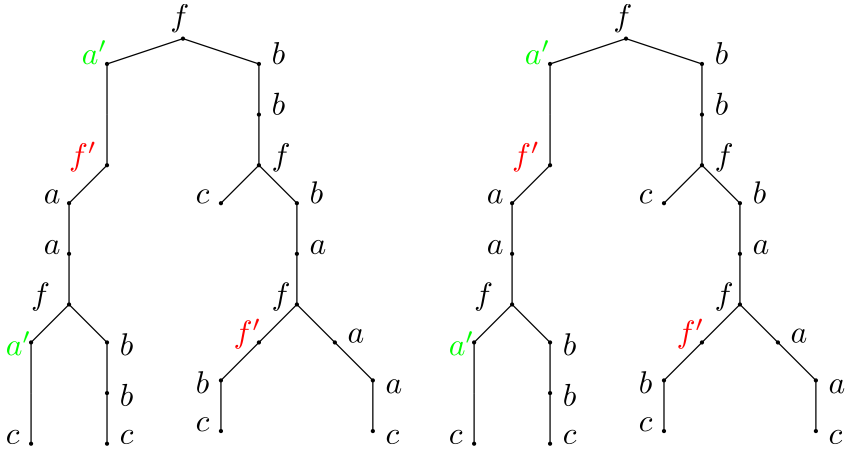

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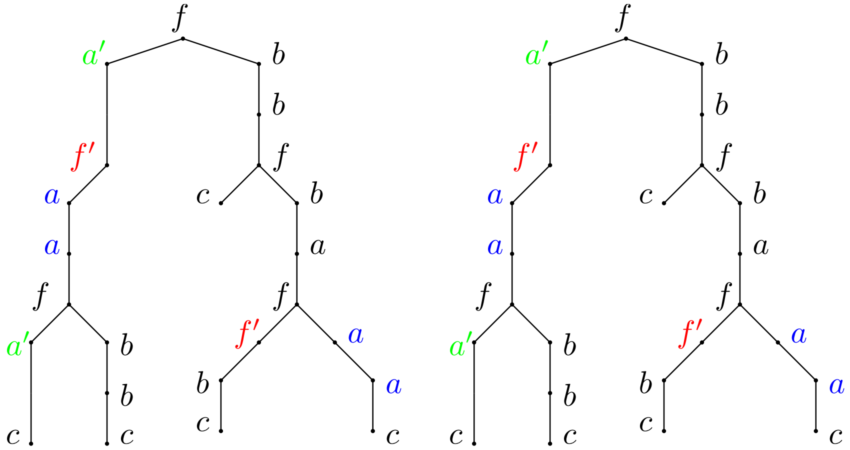

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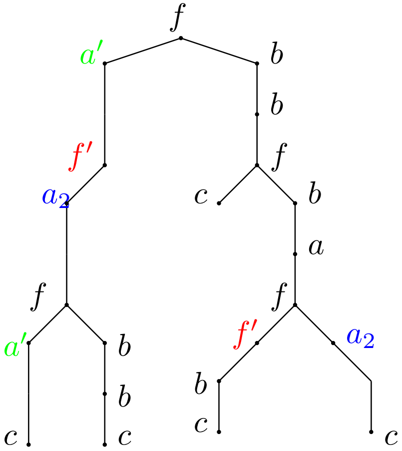

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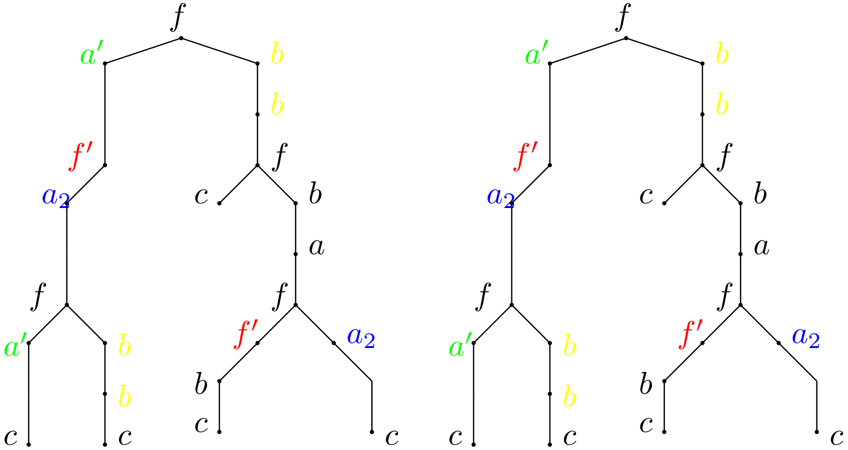

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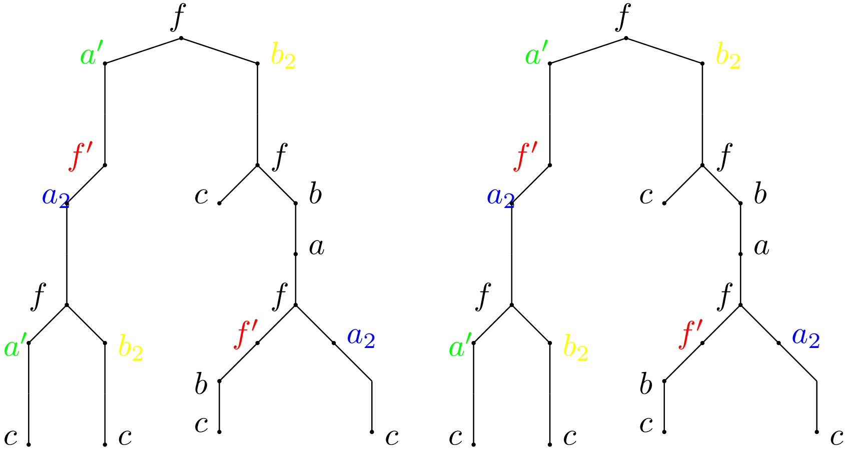

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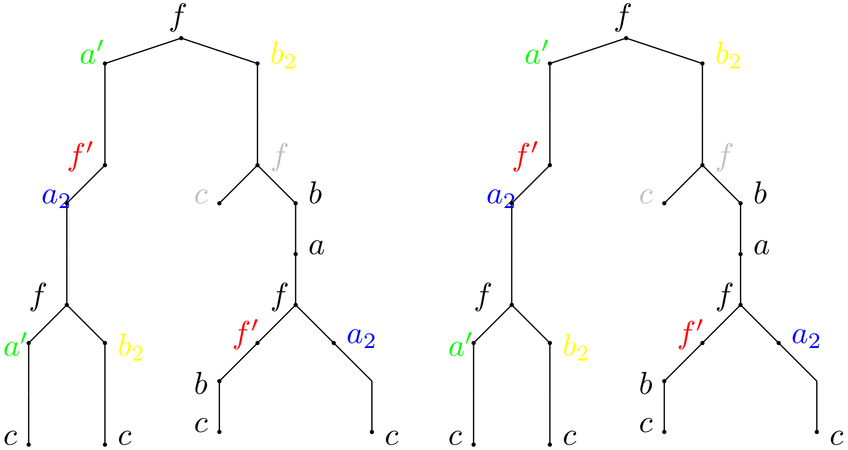

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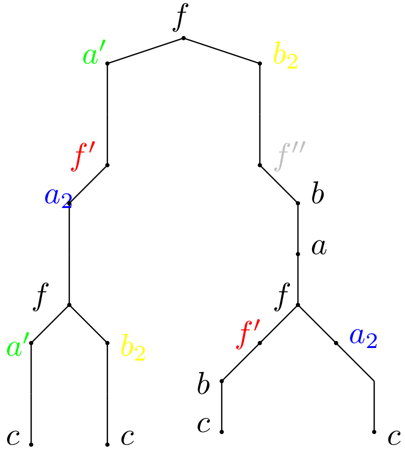

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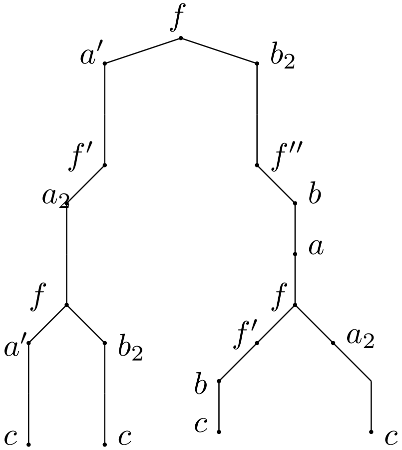

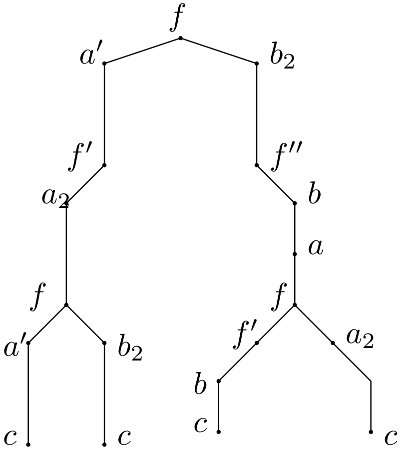

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- apply simple compression operations to the solutions
- perform them directly on the equation
  changes of the equation
- keep the size small:
  choose a compression that preserves quadratic size of the equation
- this yields PSPACE

(f, i, c)-leaf compression

- 'Absorb' i-leaf c by its father f (and change father's label to f'). Replace $f(t_1, t_2, t_3, \dots, t_{i-1}, c, t_{i+1}, \dots, t_k)$ with $f'(t_1, t_2, t_3, \dots, t_{i-1}, t_{i+1}, \dots, t_k)$

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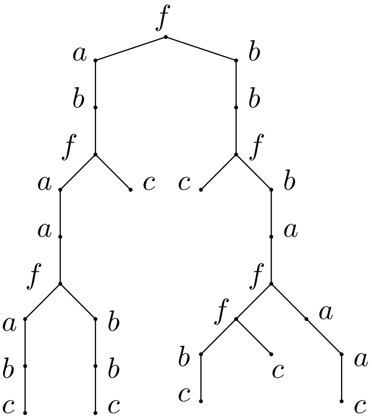

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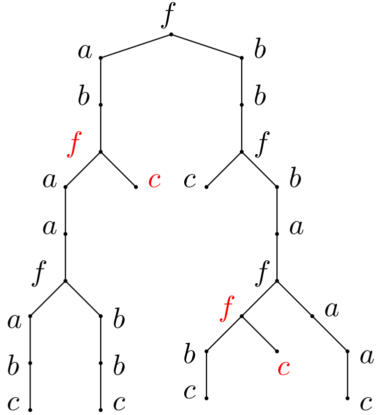

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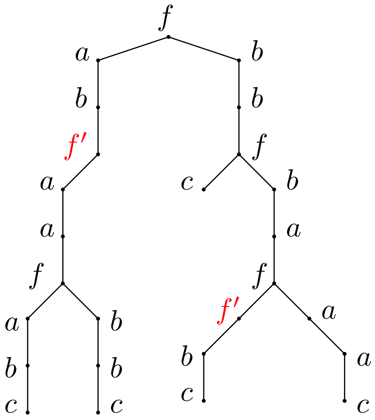

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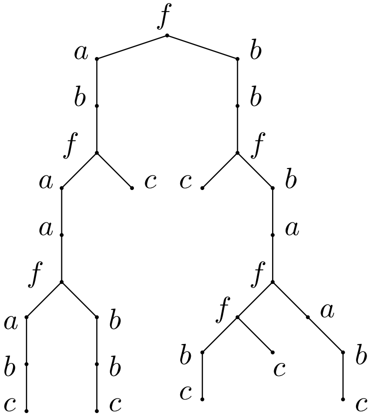

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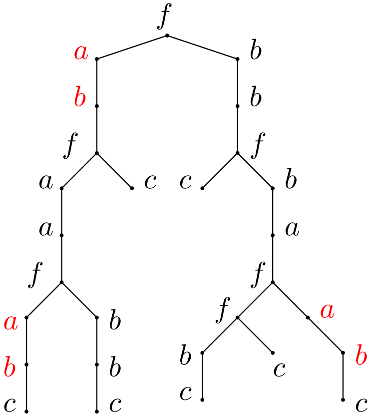

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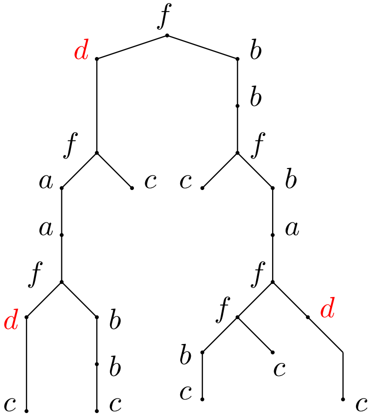

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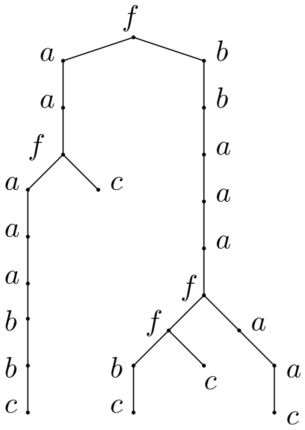

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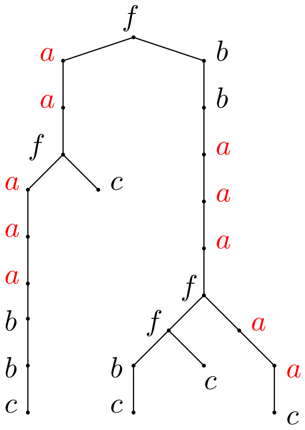

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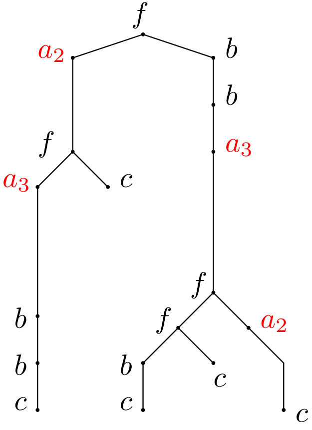


Moving to the equation.

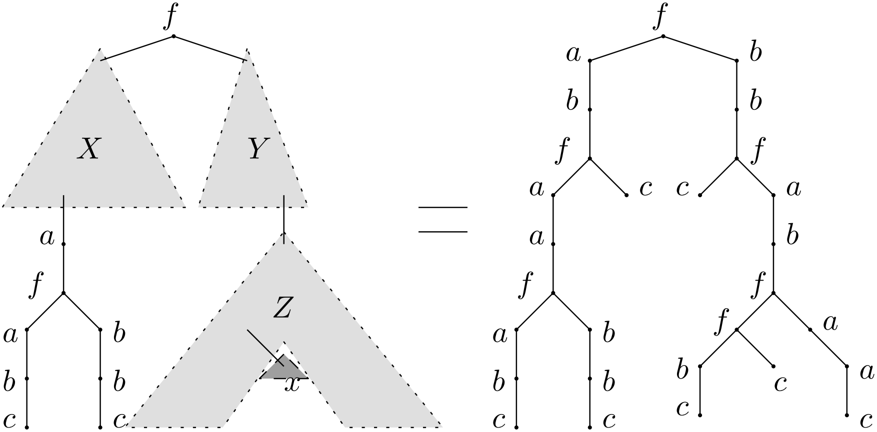
Easy case (pair compression)

Definition (pair types)

For a solution S the occurrence of ab is

- **explicit** it comes from the equation;
- **implicit** comes solely from $S(X)$ (or $S(x)$);
- **crossing** in other case.

ab is crossing (for S) if it has a crossing occurrence (for S), non-crossing (for S) otherwise.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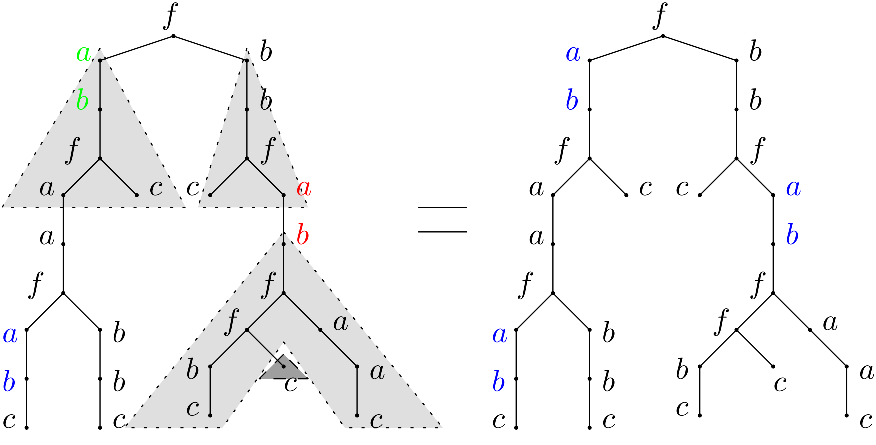

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Proof.

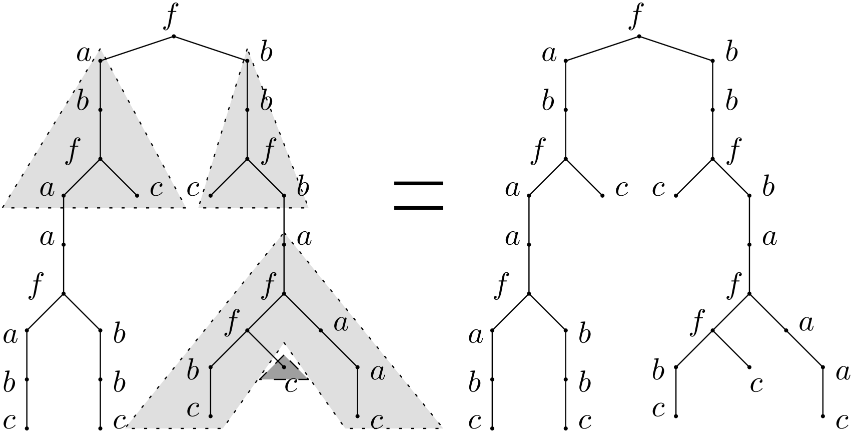
Every ab is replaced:

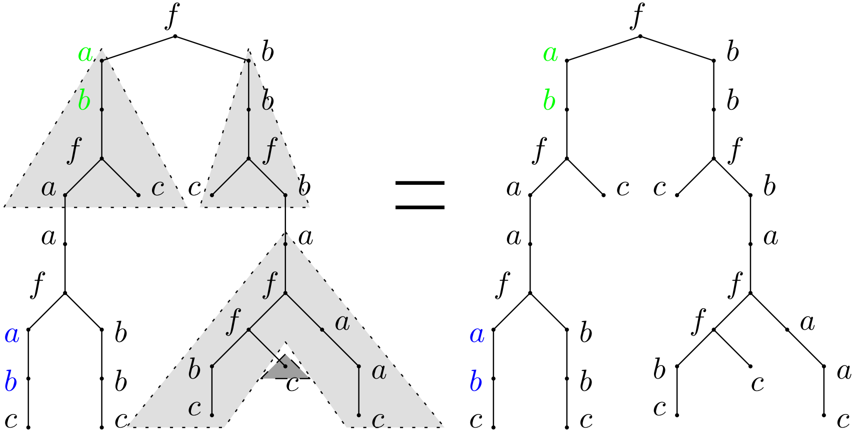
explicit pairs replaced explicitly

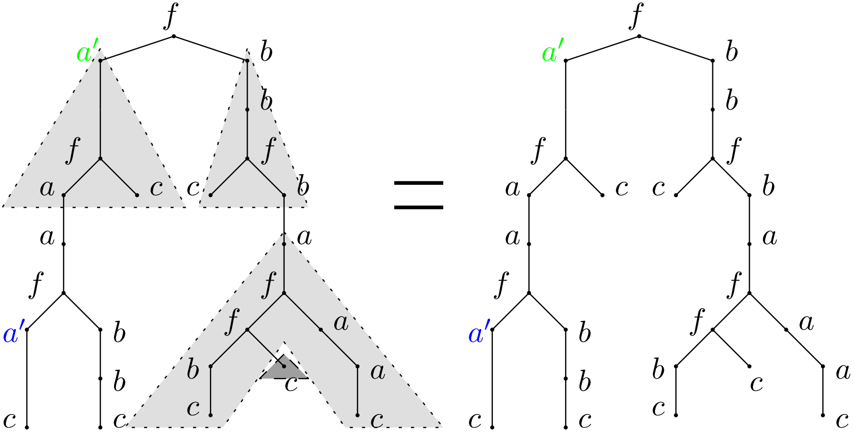
implicit pairs replaced implicitly (in the solution)

crossing there are none

$\square$







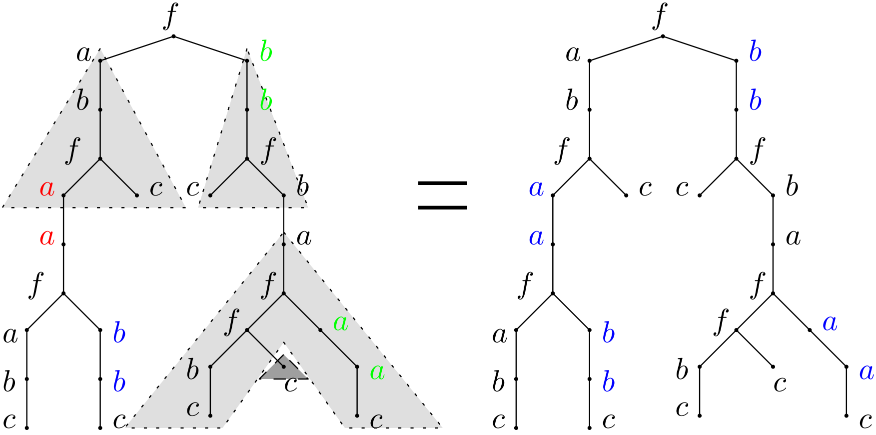
f
a′ b
b
f f
a c c b
a a
f f
f a
a′ b
b a
b c
c c c c
=
f
a′ b
b
f f
a c c b
a a
f f
f a
a′ b
b a
b c
c c
c c

Definition (a-chain types)

For a solution S the occurrence of a maximal a-chain is

- explicit: it comes from the equation;
- implicit: comes solely from $S(X)$ (or $S(x)$);
- crossing: in other case.

a has crossing chains (for S) if it has a crossing a-chain occurrence (for S), non-crossing (for S) otherwise.



ChainNCr

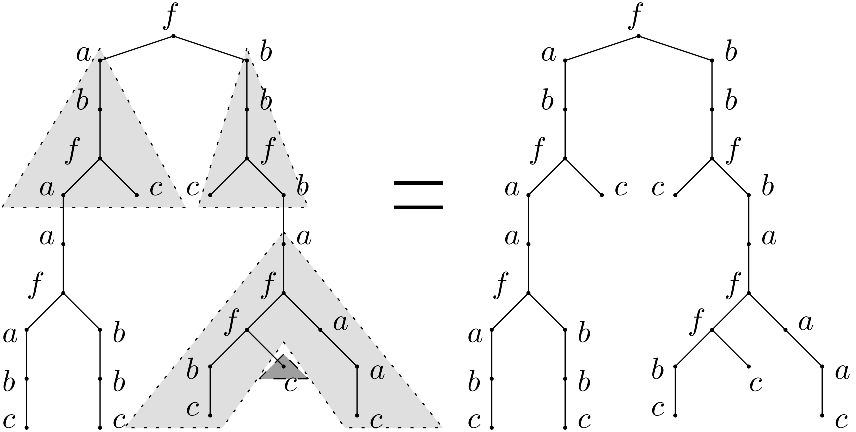
1: **for** $\ell > 1$ **do**
2: replace each explicit maximal a^ℓ in the equation by a_ℓ

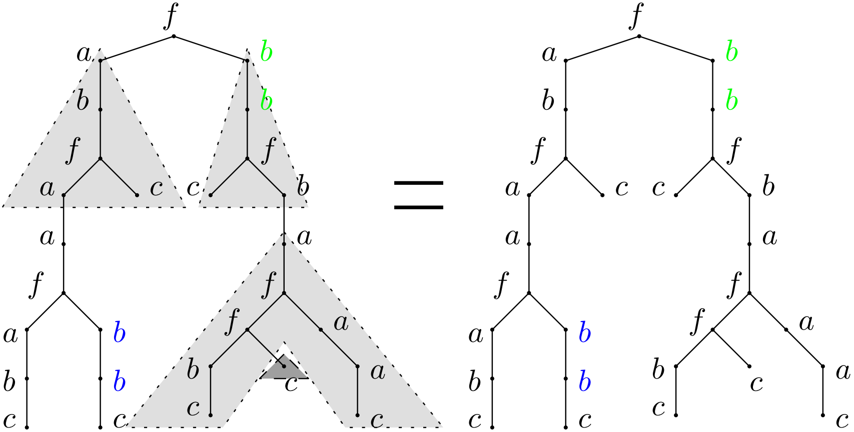
ChainNCr

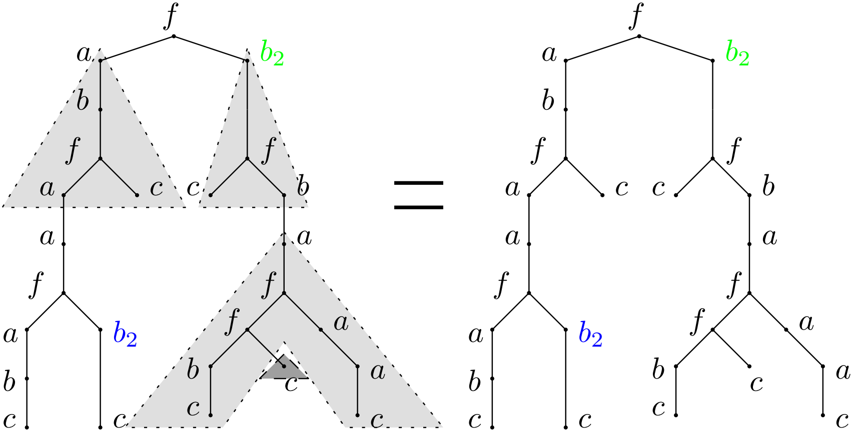
for $\ell > 1$ **do**
    replace each explicit maximal a^ℓ in the equation by a_ℓ

Lemma

ChainNCr(a) *compresses noncrossing a-chains.*






f
a
b_2
b
f
f
a
c
c
b
a
a
f
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a
b_2
f
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a
c
c
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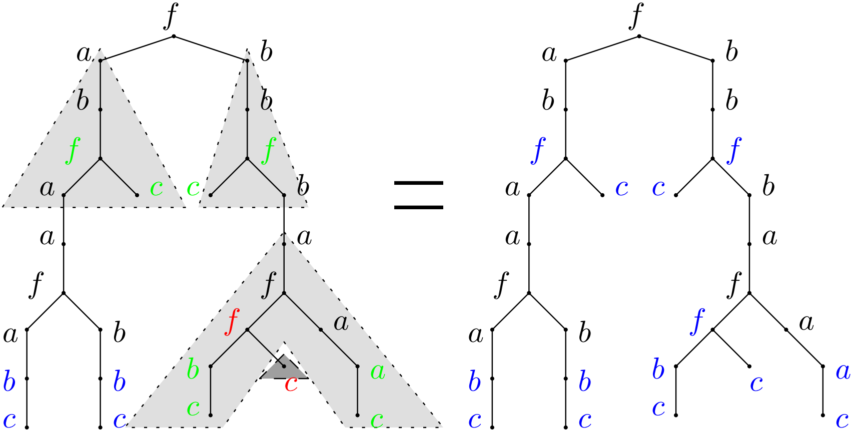
c
c
=
f
a
b_2
b
f
f
a
c
c
b
a
a
f
f
a
b_2
f
a
b
b
a
c
c
c
c
c

Definition (father-i-leaf pair)

For a solution S the occurrence of *father-i-leaf* (f, i, c) is

- **explicit** it comes from the equation;
- **implicit** comes solely from $S(X)$ (or $S(x)$);
- **crossing** in other case.

(f, i, c) is crossing (for S) if it has a crossing occurrence (for S), non-crossing (for S) otherwise.



Leaf compression ctd.

LeafNCr(f, i, c)

1: replace f with c at positions i by f'

Leaf compression ctd.

LeafNCr(f, i, c)

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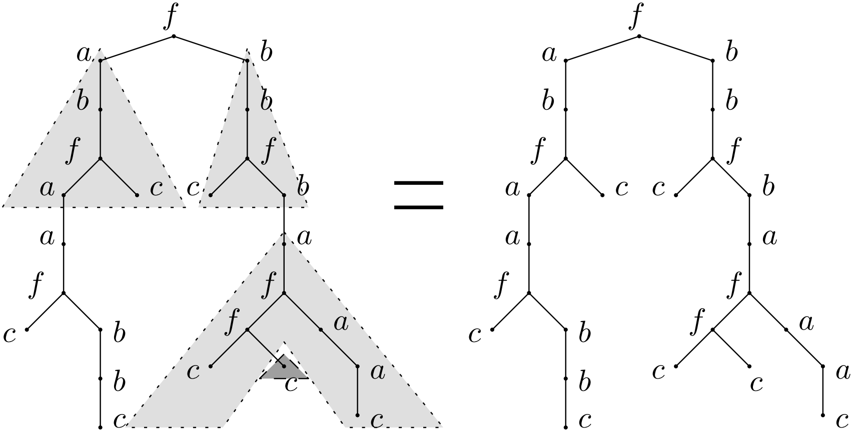
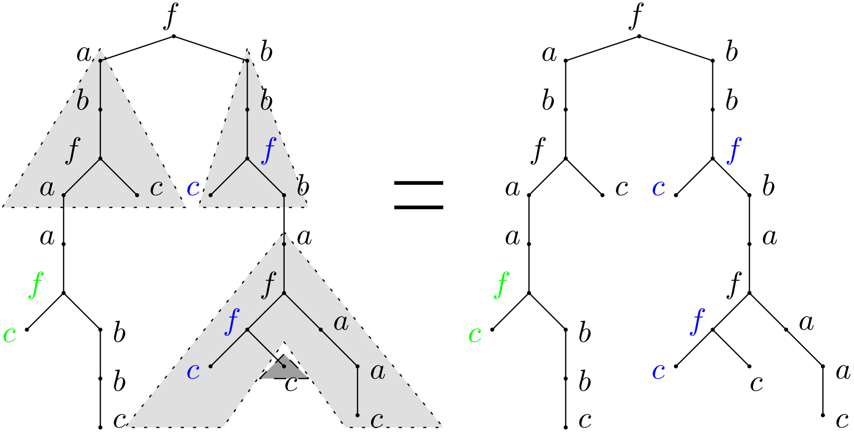
Lemma

LeafNCr *performs leaf compression when* (f, i, c) *is non-crossing.*


f f
a b a b
b b b b
f f f f
a c c b = a c c b
a a a a
f f f f
c b f a c b f a
b c c a c c a
c c c c


f
a b
b b
f f
a c c b
a a
f f
f a
c b
c a
b c
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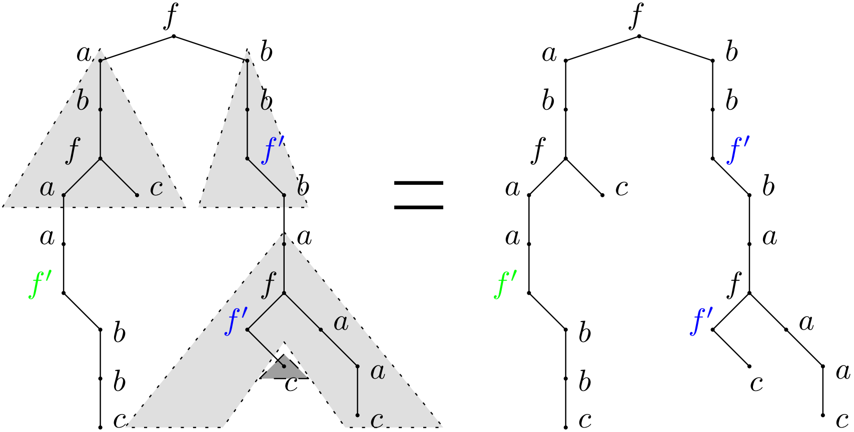
b b
f f
a c c b
a a
f f
f a
c b
c a
b c
c c



f
a
b
b
b
f
f′
a
c
b
a
a
f′
f
f′
a
b
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b
c
c
c
=
f
a
b
b
b
f
f′
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c
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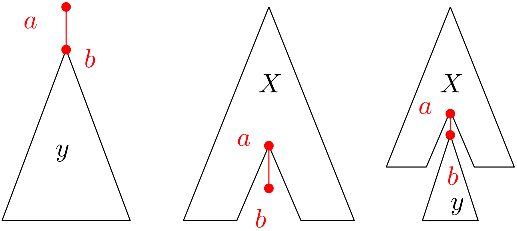
c

Uncrossing

Uncrossing pairs



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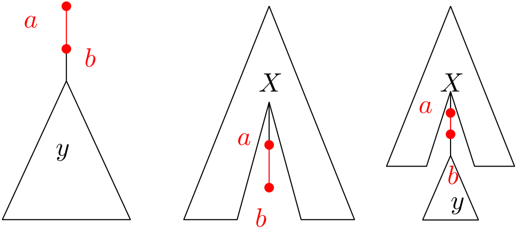

- replace Y with bY and replace X with Xa
  implicitly change $S(Y) = bt, S(X) = t'a$ to $S(Y) = t$, $S(X) = t'$

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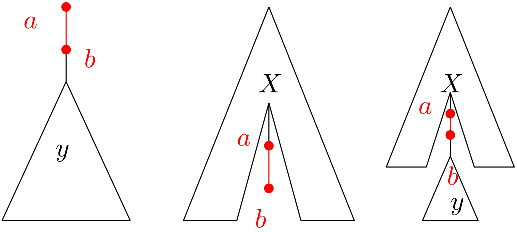

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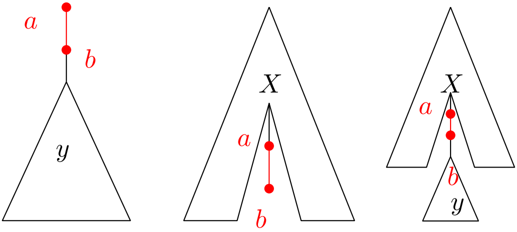

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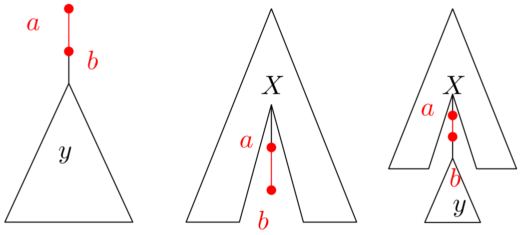

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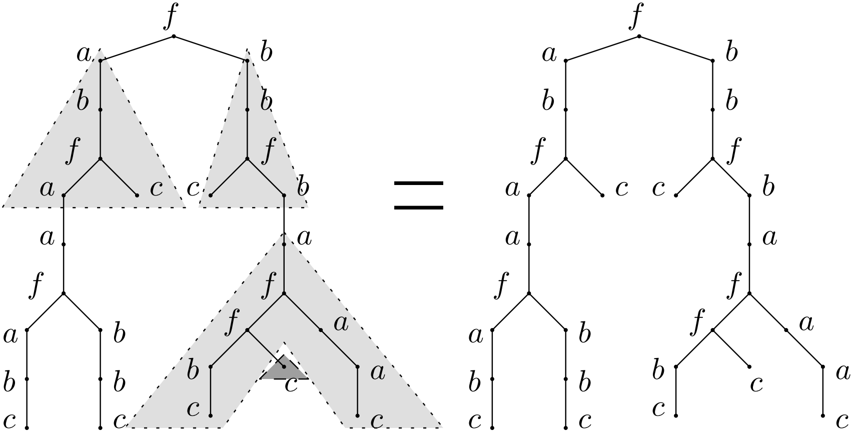
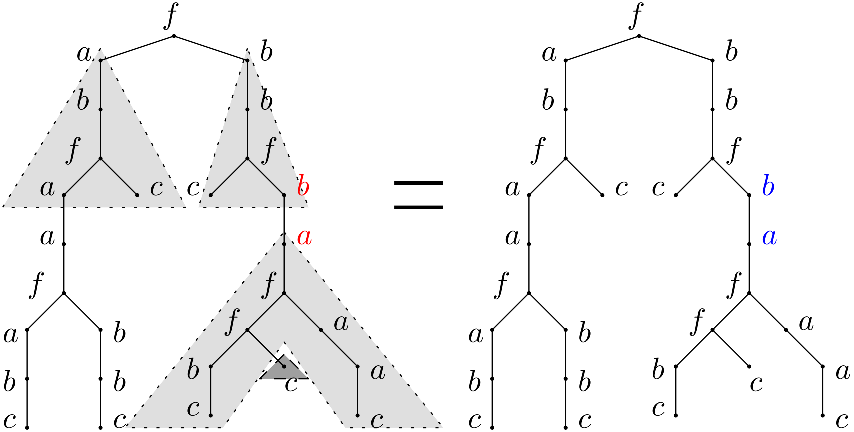
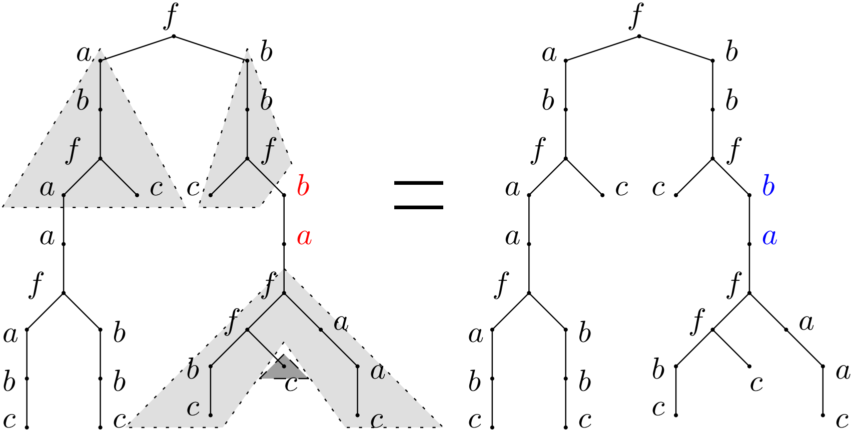
Compress it!


f f
a b a b
b b b b
f f f f
a c c b a c c b
a a a a
f f f f
a b f a a b f a
b b b a b b b a
c c c c c c c c c c
=





Uniwersytet Wrocławski

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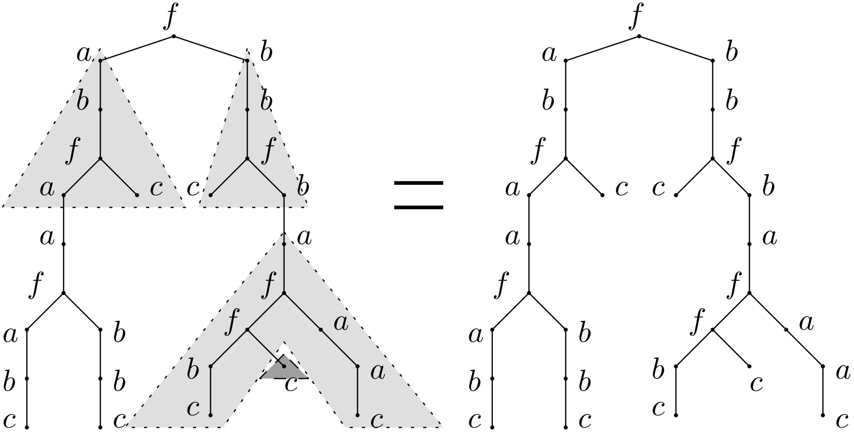
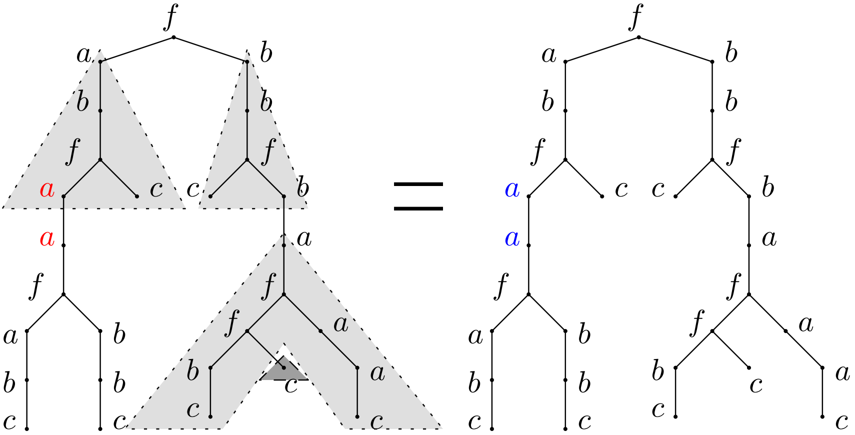
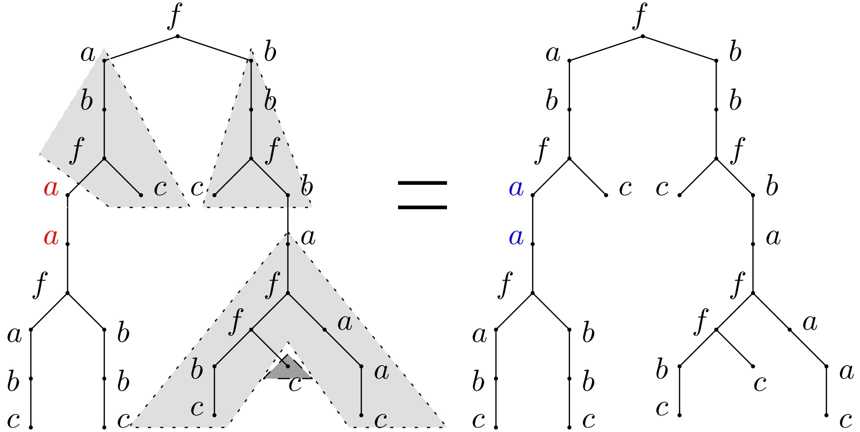
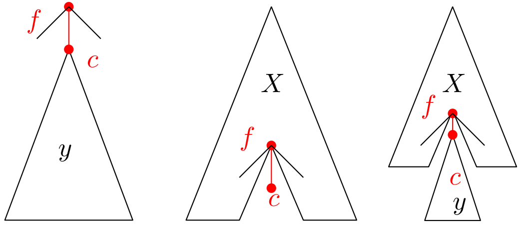
- do for all variables
- a has no crossing chains!
- compress a-chains







Uncrossing father-leaf pair



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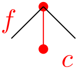

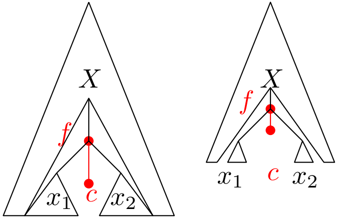

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- replace X with $X(f(x_1, \dots x_{i-1}, \bullet, x_{i+1}, \dots, x_\ell))$

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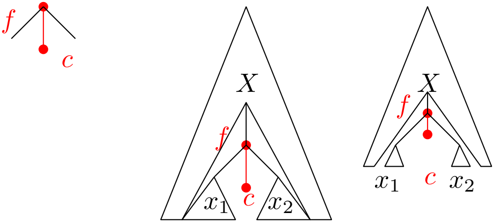

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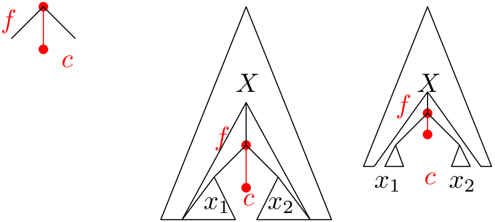

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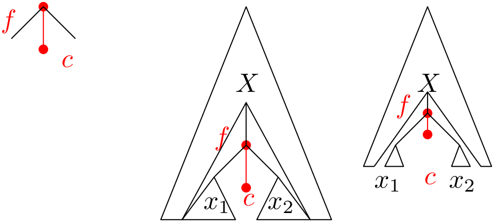

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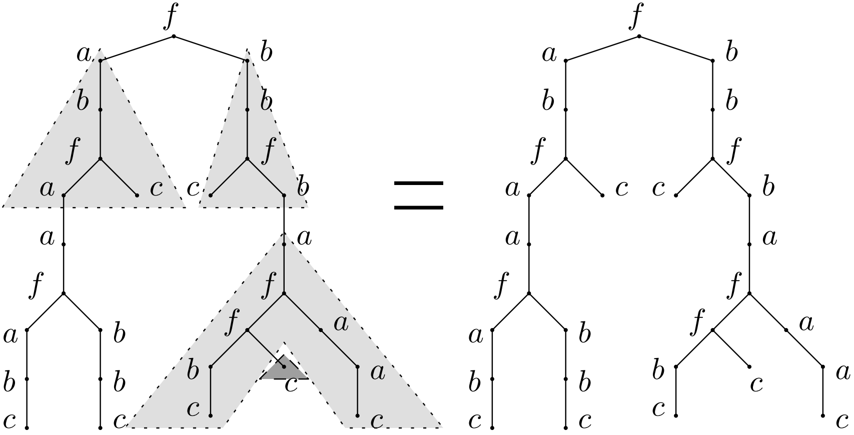
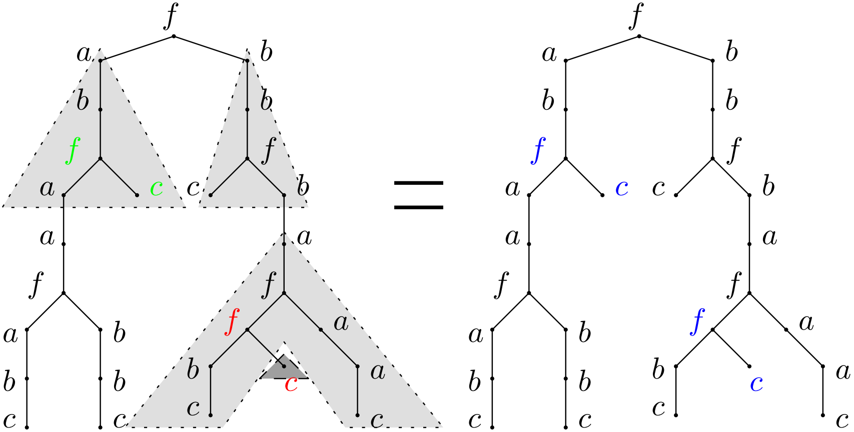
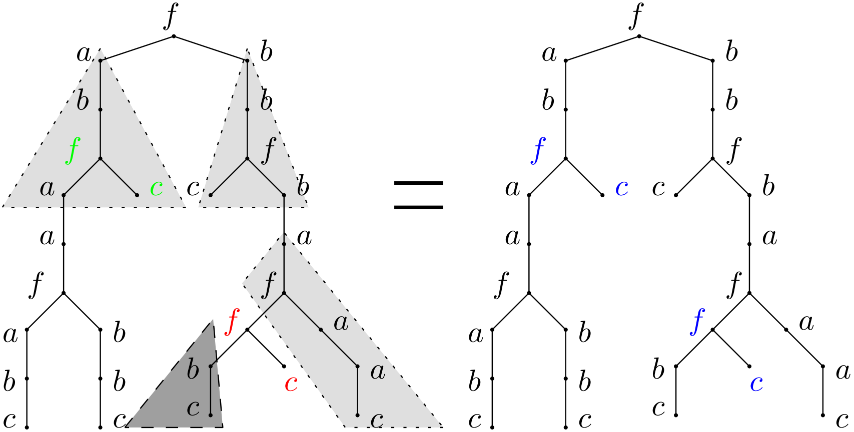
- remove X when $S(X)$ is empty

- there are no crossing (f, i, c) father-i-leaf pairs
- compress them






f
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a c c b
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a c c b
a a
f f
a b f a
b b b a
c c c c

The algorithm

while equation is nontrivial **do**

The algorithm

while equation is nontrivial **do**
    choose some ab, a or (f, i, c) to compress
    if it is crossing **then**
        uncross it
    compress it

Arity

Lemma

The maximal arity of letters in Σ does not increase.

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Compression operations do not increase arity. $\square$

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Call it k.

Controlling the variables

Lemma

There are at most

- n *context variables*
- kn *variables*

(n: size of the input equation; k—maximal arity of functions)

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(n*: size of the input equation;* k*—maximal arity of functions)*

Does not depend on the nondeterministic choices.

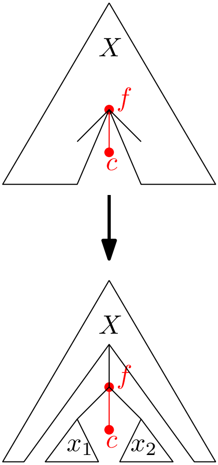
Proof.

- we do not introduce new context variables
- we can associate each 'new' variable with context variable at most $(k-1)$ are associated with one context variable $\square$

Controlling new variables: details

When to pop-down

- we pop only when needed:
  $X(c)$ and last letter of $S(X)$ is f
- at most $k-1$ new variables per X
- they are all below X.



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- $X(c')$ is in equation
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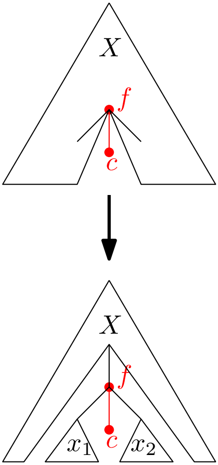

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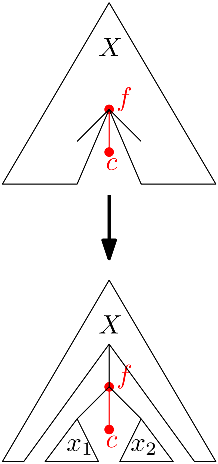

There are at most kn variables

Correctness

Soundness

If the new equation has a solution, then also the original one had.

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Just roll back the changes.

Completeness

If the equation has the solution, then for some nondeterministic choices the new equation has a corresponding one.

Make the choices according to the solution.

Size

Noncrossing

- decreases the size of the solution

Size

Noncrossing

- decreases the size of the solution
- decreases the size of the equation

Size

Noncrossing

- decreases the size of the solution
- decreases the size of the equation

Crossing

Size

Noncrossing

- decreases the size of the solution
- decreases the size of the equation

Crossing

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Size

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Size

Noncrossing

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Crossing

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Size

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We will terminate, but what about the size?

Crossing: analysis

Lemma (Fixed solution)

There are at most $kn + 2n$ different crossing letters, pairs, father-i-leaf pairs

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Lemma (Fixed solution)

Uncrossing introduces at most $kn + 2n$ letters to the equation.

Proof.

Each variable pops one up and one down
for a-chains: it is compressed immediately afterwards. □

Strategy

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We terminate for positive, never terminate for negative.

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- only crossing: only $kn + 2n$ different pairs, chains, (f, i, c)
- some covers $\frac{1}{2}\frac{m}{kn+2n}$ letters (requires some argument)
- $$m' \leq m - \frac{m}{2(kn + 2n)} + kn + 2n \implies \mathcal{O}(n^2k^2)$$

We terminate for positive, never terminate for negative.

With better analysis

- termination in $\mathcal{O}(poly(n) \log N)$ steps (N: size of the solution)
- space $\mathcal{O}(nk^2)$

Regular constraints

(Nondterministic) tree automaton

States $Q \supseteq F$: accepting states
For each $f \in \Sigma$ there is $\delta_f \subseteq Q^{\mathrm{rank}(f)+1}$ (transition function/relation)
Q-labelling consistent with transition functions
Accepting: root label in F

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t with k arguments has transition $\delta_t \subseteq Q^{k+1}$:
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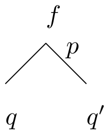
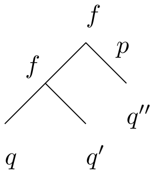


$\delta_{f(\cdot,\cdot)} =$
$\{(q, q', p) \ : \ p \in \{q+1, q'\}\}$



$\delta_{f(f(\cdot,\cdot),\cdot)} =$
$\{(q, q', q'', p) \ : \ p \in \{q+2, q'+1, q''\}\}$

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Constraints: given automaton N declare

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Extend the algorithm.

Extensions

Easy

Compression: When a, b are compressed: $\delta_c \leftarrow \delta_a \circ \delta_b$
Popping: when x pops a: set δ'_x so that $\delta_x = \delta_a \circ \delta'_x$

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Less easy

- Size of the transition function?
- Size of the signature?
  We cannot trim—this affects transition function.

Transition functions

Each needed transition δ is in one of the forms:

- $\delta \in Q$, so with 0 parameters (not so many)
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- Σ is stored using an EXPTIME oracle (verified at popping)
  - EXPTIME: exponentially many different transition vectors
  - sometimes the oracle is simpler (below EXPTIME)

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- Fragment with one context variable (in P?)