Context-free word problem semigroups

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Word problem for groups

Theorem (Anīsīmov 1971)

A group has regular word problem if and only if it is finite.

Theorem (Muller & Schupp 1983 & Dunwoody 1985)

A group has context-free word problem if and only if it is finitely generated and virtually free.

Group Set with associative binary operation, identity (neutral) element, inverses.

Word problem For a group G with respect to finite set of generators A

$$WP(G, A) = \{ w \in A^* : w =_G 1_G \}.$$

Free group Set of reduced words over $\{ a, a^{-1} : a \in A \}$.

Virtually free Group with a finite-index free subgroup.

Word problem for groups

Deciding equality

- Two words $u = u_1 u_2 \cdots u_k, v = v_1 v_2 \cdots v_\ell \in A^*$ are equal in G if and only if $uv^{-1} = u_1 u_2 \cdots u_k v_\ell^{-1} \cdots v_2^{-1} v_1^{-1} \in \mathrm{WP}(G, A)$.

Proposition

1. Having context-free word problem is invariant under change of (finite) generating set.
2. If a group has context-free word problem, so does all of its finitely-generated subgroups.

Word problem for semigroups and monoids

Semigroup Set with associative binary operation.

Monoids Set with associative binary operation, identity element.

In groups: $\begin{cases} WP(G,A) = \{ w \in A^* : w =_G 1_G \}; \\ u =_G v \iff uv^{-1} \in \mathrm{WP}(G,A). \end{cases}$

Let S be a semigroup, A a generating set for S. The word problem for S is the language

$$WP(S,A) = \{ u\#v^{\mathrm{rev}} : u, v \in A^+, u =_S v \}.$$

Example

The word problem of the free semigroup A^+ over A is

$$\mathrm{WP}(A^+, A) = \{ u\#u^{\mathrm{rev}} : u \in A^+ \};$$

this is context-free but not regular.

Proposition

A semigroup has regular word problem if and only if it is finite.

Previous results

Theorem (Hoffman, Holt, Owens, Thomas, DLT 2012)

Let S be a semigroup, T a subsemigroup, with $S \setminus T$ finite. Then S has context-free word problem if and only if T has context-free word problem.

Theorem (Hoffman, Holt, Owens, Thomas DLT 2012)

Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a finitely generated completely simple semigroup. Then S has a context-free word problem if and only if G has a context-free word problem.

$\mathcal{M}[G; I, \Lambda; P]$ Copies of a group G in a $I \times \Lambda$ array; 'close' to being a group.

Inverse monoids

Inverse monoid For all $u \in M$ there exists a unique $u^{-1} \in M$ with $uu^{-1}u = u$ and $u^{-1}uu^{-1} = u^{-1}$.

- Standard example: partial bijections under composition.
- Symmetries of quasi-crystals.

Proposition (Brough 2018)

Free inverse monoids do not have context-free word problem.

- Straightforward application of pumping lemma for context-free language.

Context-free word problem semigroups

Theorem (C. & Maltcev 2012)

Let $(\mathcal{A}, \mathcal{R})$ be a confluent context-free monadic rewriting system. Then the monoid presented by $\langle \mathcal{A} \mid \mathcal{R} \rangle$ has context-free word problem.

Confluent Choices in rewriting do not matter.

Monadic Right-hand side of rewriting rules have length at most 1.

Context-free For each possible right-hand side, the set of left-hand sides is a context-free language.

Deterministic context-free word problem

Theorem (Muller & Schupp 1983 & Dunwoody 1985)

For groups, the following are equivalent:

- being finitely generated and virtually free;
- having context-free word problem;
- having deterministic context-free word problem.

Example (Brough, C., Pfeiffer)

Let $S = \langle a, b, x, y, z \mid (xww^{\text{rev}}y, z) : w \in \{a, b\} \rangle$. Then:

- has context-free word problem (by the rewriting systems result)
- does not have deterministic context-free word problem (since the language of palindromes is not deterministic context-free).

Deterministic context-free word problem

Conjecture

The bicyclic monoid has context-free word problem (by the rewriting systems result); conjecture it does not have deterministic context-free word problem.

Bicyclic monoid $\langle b, c \mid (bc, \varepsilon) \rangle$; inverse monoid; 'easiest' monoid that is not a group (all proper homomorphic images are groups).

Classification & Constructions

Question

Can we classify

- semigroups that have context-free/deterministic context-free word problem?
- semigroups of particular types that have context-free/deterministic context-free word problem?

Question

What can be said about the interaction of the classes of semigroups with context-free/deterministic context-free word problem and semigroup constructions (direct products, free products, . . .)?

- Is the class closed under a given construction?
- If not, can we classify when the result is in the class?
- Is the class closed under regressing from the result of a such a construction to the original semigroup(s)?

Results

Classification/construction result summary (Brough, C., Pfeiffer)

- Classification of direct products that have context-free word problem.

- Closure of class of semigroups with context-free word problem under passing to/from free products.
- Closure of class of semigroups/monoids with context-free word problem under passing to and from Rees matrix semigroups.
- Classification of strong semilattices of semigroups with context-free word problem.
- Classification of Bruck–Reilly extensions with context-free word problem.

Direct products

Direct product $S \times T$ under componentwise multiplication

Corollary (of Muller–Schupp Theorem)

$G \times H$ has context-free word problem if and only if one of G, H is finite and the other is virtually free.

Theorem (Brough, C., Pfeiffer)

$S \times T$ has context-free word problem if and only if it is either:

- finite, or
- S has context-free word problem and T is finite and all elements of T are decomposable, or vice versa.

Decomposable For $x \in T$, there exist $p, q \in T$ such that $pq = x$.

Direct products and finite generation

If G and H are f.g. monoids/groups, $G \times H$ is always finitely generated.

Not true for general semigroups:

- Let $T = \{z, 0\}$ with all products 0.
- All elements (x^α, z) in $\{x\}^+ \times T$ are indecomposable.
- So $\{x\}^+ \times T$ is not finitely generated.

Theorem (Robertson, Ruškuc, Wiegold 1998)

1. Let S and T be infinite. Then $S \times T$ is finitely generated if and only if S and T are finitely generated and all elements of S and T are decomposable.
2. Let S be infinite and T finite. Then $S \times T$ is finitely generated if and only if S is finitely generated and all elements of T are decomposable.

Direct products

Difficult part to prove

$S \times T$ is infinite and has context-free word problem
$\implies S$ has context-free word problem and T is finite.

Idea of proof.

- Suppose $S \times T$ is infinite and has context-free word problem.
- Careful choice of generating set A
  + technical stuff
  + taking homomorphic images shows that
  S and T have context-free word problem.
- Build a PDA recognizing word problem of $S^1 \times T^1$ with respect to
$$A_S \cup A_T \cup A.$$
- Intersect with
$$A_S^* A_T^* \# A_S^* A_T^*.$$
- Pumping lemma shows that one of S^1, T^1 is finite (assume T^1).

□

Open problems

Question

Does there exist an infinite periodic semigroup with context-free word problem?

Periodic For all $x \in S$, there exist $\ell < m$ with $x^\ell = x^m$.

Problem

Understand better the class of semigroups with deterministic context-free word problem in terms of closure/classification results.