First lower bounds for palindromic length

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Palindromic length

The palindromic length $PL(u)$ is the minimal number of concatenated palindromes needed to express the word u.

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The prefix palindromic length $PPL_u(n)$ of an infinite word u is the palindromic length of the prefix of length n of u.

Conjecture

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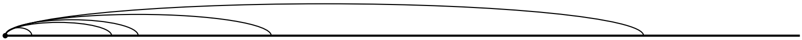
The proof is not constructive
The number of palindromes starting at the same point is logarithmical.



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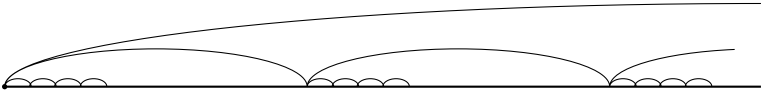
The non-constructive idea is in fact a bit more general, so, the remaining case is when there is an unbounded number of layers of unbounded powers.



Is it an exotic situation?

In this formulation, not: almost all words are like this. But due to [Saarela 2017], only the (rare) case when the word does not contain long words other than that, remains unsolved.

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So, in fact, the conjecture is proven for almost all infinite words.

What are basic examples of the “bad” case?

Sturmian words

$$\begin{aligned} s_{-1} &= b \\ s_0 &= a \\ s_{n+1} &= s_n^{d_n} s_{n-1} \text{ for all } n \geq 0. \end{aligned}$$

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d_n are unbounded $\iff$ "bad" case, [F. 2018]

Sturmian example

$\mathbf{d} = (2, 5, 1, \cdots)$

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s_{-1} &= b \\
s_0 &= a \\
s_1 &= aab \\
s_2 &= aabaabaabaabaaba \\
s_3 &= aabaabaabaabaaba\ aab
\end{aligned}
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etc.

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The palindromic length conjecture is proven for all Sturmian words, but the proofs are completely different for bounded and unbounded $\mathbf{d}$.

Simple Toeplitz examples

$$\varphi_d : \begin{cases} a \to a^{d-1}b, \\ b \to a^d. \end{cases}$$

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Example: $\mathbf{d} = (3, 5, \cdots)$

$$v_{\mathbf{d}} = aab\ aab\ aab\ aab\ aaa\ \ aab\ aab \cdots$$

Toeplitz result

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The prefix palindromic length of $v_{\mathbf{d}}$ is unbounded for every $\mathbf{d}$ with an infinite number of $d_i > 2$.

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The prefix palindromic length of $v_{\mathbf{d}}$ is unbounded for every $\mathbf{d}$ with an infinite number of $d_i > 2$.

Sketch of the proof. For simplicity, consider $d_1 = d_2 = \cdots = d > 2$.

The factor $v_{\mathbf{d}}[i+1] \cdots v_{\mathbf{d}}[j]$ is a palindrome iff in the d-ary representation,

- either $i = x_l \cdots x_m \cdots x_0$ and $j = x_l \cdots x_{m+1} y_m \overline{x_{m-1}} \cdots \overline{x_0}$, where $\overline{x} = d - 1 - x$;
- or $i = \boxed{s.d^{m+2}} + (d-1)x_m \cdots x_0$ and $j = \boxed{(s+1)d^{m+2}} + 0y_m\overline{x_{m-1}} \cdots \overline{x_0}$.

Example

$$d = 4$$

$$v = aaab\ aaab\ aaab\ aaaa\ \ aaab\ aaab \cdots .$$

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$v(12..19]$: $12 = 030_4$, $19 = 103_4$ (both types)

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Proposition

The number of digits not equal to 0 or $d-1$ in the d-representation of i and j differ by at most 2.

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$$PPL_v(\underbrace{1\cdots 1}_{2k-1}) \geq k.$$

- This works for every sequence $\mathbf{d}$ with $d_i > 2$,
- but not for $d_1 = d_2 = \cdots = 2$ (the period doubling word).

Little open problem

Give a formula or a lower bound for $PPL(n)$ or the period-doubling word

$$p = ab\ aa\ abab\ abaaabaa\ abaaabab \cdots$$

generated by

$$\delta : \begin{cases} a \to ab, \\ b \to aa. \end{cases}$$

Observation

For the remaining case of the main conjecture, some (sophisticated) numeration systems can be useful.

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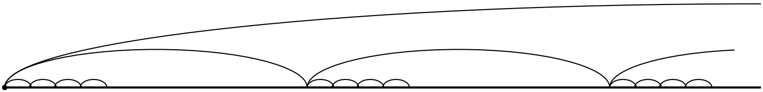

Thue-Morse word

$$\tau : \begin{cases} a \to abba, \\ b \to baab. \end{cases}$$

$$t = \tau(t) = abba\ baab\ baab\ abba\ baab\ abba \cdots$$

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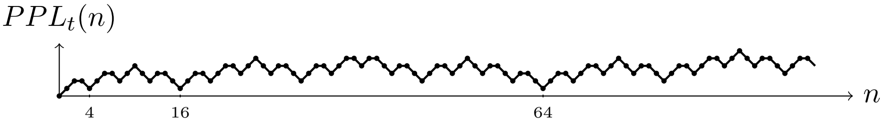

Thue-Morse word

Theorem

The following identities hold for all $n \geq 0$*:*

$$\begin{aligned} PPL_t(4n) &= PPL_t(n), \\ PPL_t(4n+1) &= PPL_t(n)+1, \\ PPL_t(4n+2) &= \min(PPL_t(n), PPL_t(n+1))+2, \\ PPL_t(4n+3) &= PPL_t(n+1)+1. \end{aligned}$$

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Formula rediscovered by Shuo LI, 29/07/2019

Morphism for first differences

$$d_t(n) = PPL_t(n+1) - PPL_t(n) \in \{\searrow, \rightarrow, \nearrow\}$$

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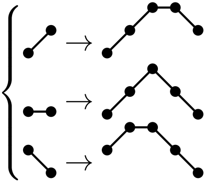
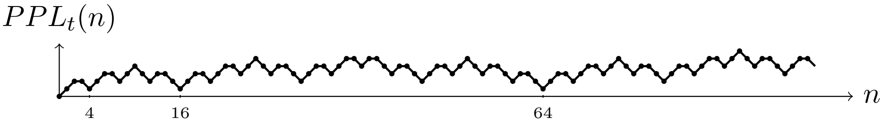
Proposition

The sequence $(d_t(n))$ *is the fixed point of the morphism*

$\delta :$ 

Growth



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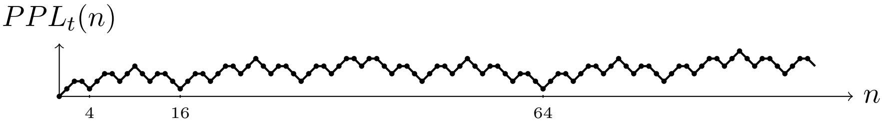

$$\limsup \frac{PPL_t(n)}{\ln n} = \frac{3}{4\ln 2}.$$

k-regularity?

For the Thue-Morse word, the sequence $PPL_t(n)$ is 4-regular

(meaning that $PPL_t(n)$ can be computed directly from the 4-decomposition of n).

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(meaning that $PPL_t(n)$ can be computed directly from the 4-decomposition of n).

Is $PPL_u(n)$ a k-regular sequence if u is k-automatic?