Uniwersytet Wrocławski

# Recompression for Word Equations

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University of Wrocław
Developments in Language Theory
Warsaw 08.08.2019

# DLT: always first

DLT 2007 my first conference

DLT 2013 my first invited talk

DLT 2019 my first life-achievement award

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Systems of equations $\{U_i = V_i\}_{i \in I}$, where $U_i, V_i \in (\Sigma \cup \mathcal{X})^*$.
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- nontriviality of solutions
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**Wrong**

Decidable [Makanin '77].

# Satisfiability and word combinatorics

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Heavily based on word combinatorics.

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- regular constraints
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... but no breakthrough.

# Compression and word equations

## Plandowski & Rytter

A length-minimal solution of size $N$ a word equation of size $n$ has an SLP of size $poly(n, \log N)$.

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Guess compressed representation of variables and verify the guess. Depends on external guarantees for $N$ (Makanin's algorithm).

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## Remark

Compress and compute paradigm turns out to be useful all around.

# Plandowski & Rytter: Idea

## Cut

$S : \mathcal{X} \to \Sigma^+$, extend to $S : (\mathcal{X} \cup \Sigma)^+ \to \Sigma^+$

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## Main observation

If $S$ is length-minimal and $w$ occurs in $S(u)$, then it "touches" a cut. (i.e. the cut is inside or at one of the ends)

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## Main observation

If $S$ is length-minimal and $w$ occurs in $S(u)$, then it "touches" a cut. (i.e. the cut is inside or at one of the ends)
Otherwise the word could be removed.

# Plandowski & Rytter proof

For a cut $\alpha$ let $(\alpha)_k$: the word $2^{k-1}$ to the left and right of the cut.

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$$\overbrace{|a|a\ a\ a|a\ a\ a|a\ a\ a|}^{(\alpha)_3}\ |a\ a\ a|a|a\ a|a\ a|a\ a|$$

$$a\quad X\quad X\quad X\ =\ X\ a\ Y\ Y\ Y$$

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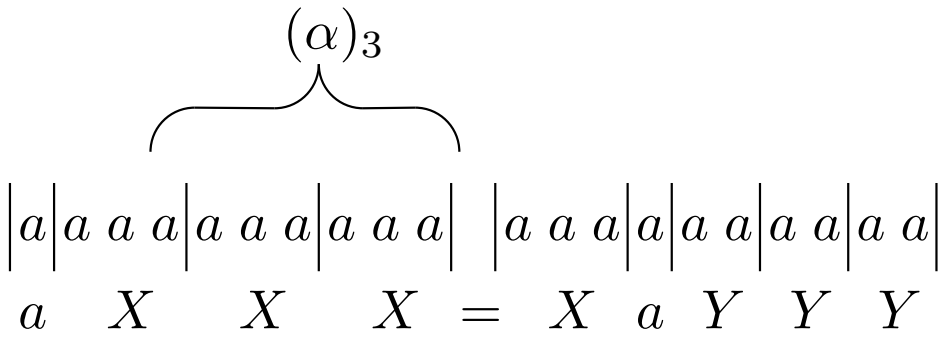


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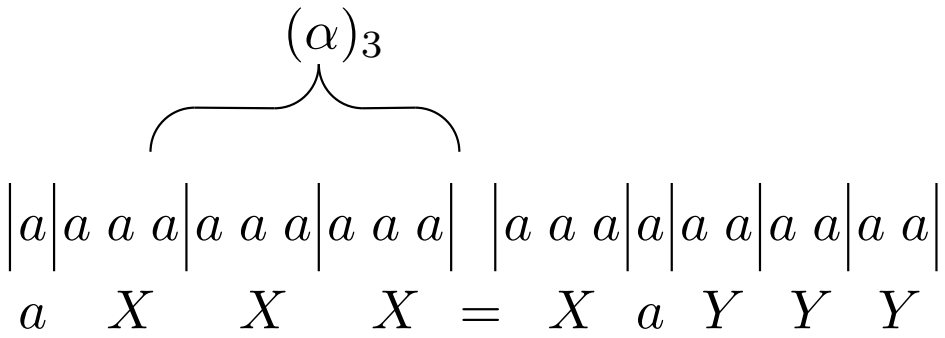


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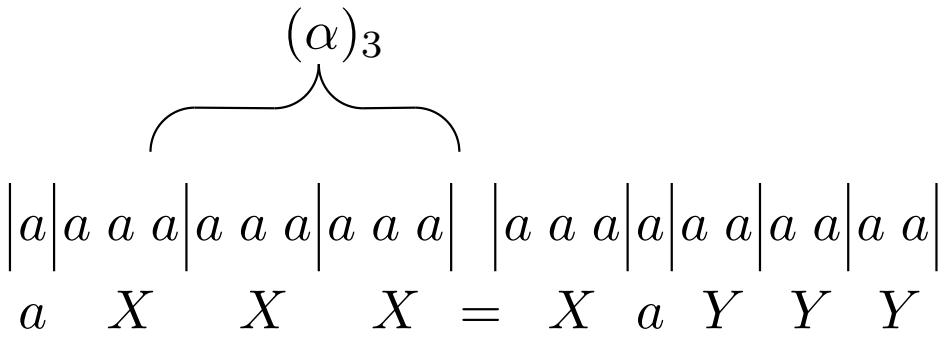


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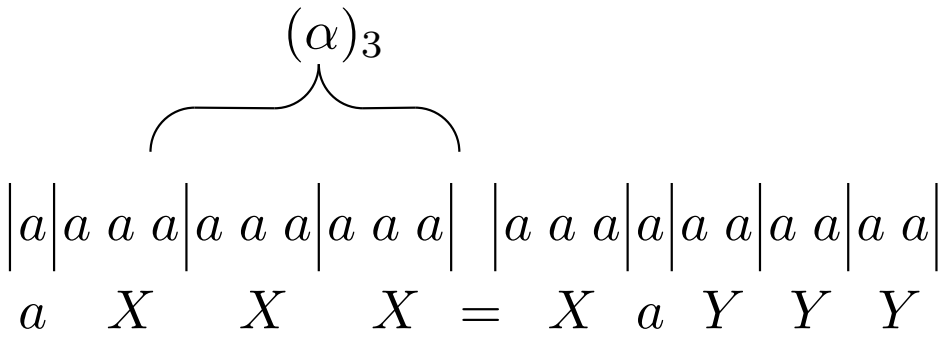


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Almost an SLP: can be transformed (quadratic size increase).
Size $((|U| + |V|)\log N)^2$.

# Towards PSPACE

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*Length-minimal solution is at most doubly exponential.*

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**Theorem (Plandowski 1999)**

*Length-minimal solution is at most doubly exponential.*
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**Theorem (Plandowski 1999)**

*Satisfiability of word equations is in PSPACE.*
*Via: even better factorization and merging together.*

# What next

State of the art for 20 years.
Extended (free groups, regular constraints, representation of all solutions...)

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## In the retrospect

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## Why not bottom-up?

There has to be a rule $X \to ab$.
We can build SLP bottom-up.

# Building SLP bottom-up

Build SLP bottom-up for the solution.

- choose a pair $ab$ and replace it in the solution word
  There are only $\mathcal{O}(|U| + |V|)$ of them.

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I read it later and realized it can be done simpler.

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- A solution word for length-minimal solution
- factorization: chosen substrings + letters—factors (size: number of factors)
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- SLP size: $\mathcal{O}(p(|U|+|V|)\log N)$.

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- Proceed iteratively
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$$Xba Yb = baaababbbab \quad \text{for } S(X) = baa \; S(Y) = bba$$
$$X\,c\,Yb = c\,aa\,c\,b\,c\,b \quad \text{for } S'(X) = caa \;\; S'(Y) = bc$$

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And what about replacing $ab$ by $d$?

$$XbaYb = baaababbbab \qquad \text{for } S(X) = baaa\ S(Y) = bba$$

# Pair types

## Definition (Pair types)

Occurrence of $ab$, $a \neq b$, in a solution word (so for a fixed solution) is

- **explicit** it comes from $U$ or $V$;
- **implicit** comes solely from $S(X)$;
- **crossing** in other case (crosses a cut).

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**Lemma**

*The* PairComp$(a, b)$ *properly compresses noncrossing pairs.*

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complete if the old equation has a solution then the new one has

sound if the new equation has a solution then the old one has

# Correctness.

## Complete

$S'(U')$ is $S(U)$ with every $ab$ replaced; similarly $S'(V')$:

explicit pairs replaced explicitly

implicit pairs replaced implicitly (in the solution)

crossing there are none

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$$c\,aa\;\; c\;ab\;c\;b = c\;aa\;c\;ab\;c\;b$$
$$X\;\;\; c\;a\;Y\;b = c\;aa\;c\;ab\;c\;b \qquad S'(X) = caa\;S'(Y) = bc$$

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$$S(X) = baaa \; S(Y) = bba$$

$$c\,aa \;\; c\,ab\,c\,b = c\,aa\,c\,ab\,c\,b$$

$$X \;\; c\,a\,Y\,b = c\,aa\,c\,ab\,c\,b \qquad S'(X) = caa \; S'(Y) = bc$$

# Correctness.

## Complete

$S'(U')$ is $S(U)$ with every $ab$ replaced; similarly $S'(V')$:

explicit pairs replaced explicitly

implicit pairs replaced implicitly (in the solution)

crossing there are none

## Sound

If the new equation is satisfiable: roll back the changes.

$$X\ baa\ Y\ b{=}baaabaabbbab \qquad S(X) = baaa\ S(Y) = bba$$
$$baaabaabbbab{=}baaabaabbbab$$
$$c\,aa\ \ c\,ab\,c\,b{=}\ c\,aa\,c\,ab\,c\,b$$
$$X\ \ c\,a\,Y\ b{=}\ c\,aa\,c\,ab\,c\,b \qquad S'(X) = caa\ S'(Y) = bc$$

# Dealing with crossing pairs

$ab$ is a crossing pair

There is $X$ such that $S(X) = bw$ and $aX$ occurs in $U = V$ (or symmetric).

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## Uncrossing$(a, b)$

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2: **if** first letter of $S(X)$ is $b$ **then**
3: replace each occurrence of $X$ by $bX$ ▷ Pop
▷ Change $S$ accordingly
4: **if** $S(X) = \epsilon$ **then** remove $X$ from the equation
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We can compress it.

# Uncrossing: example

## Uncrossing $ab$

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## Lemma (Length-minimal solutions)

*If $a^\ell$ is a maximal block in a length-minimal solution of $u = v$ then $\ell \leq 2^{c|uv|}$.*

# Blocks compression

## When $a$ has no crossing block

1: **for** all maximal blocks $a^\ell$ of $a$ and $\ell > 1$ **do**
2: let $a_\ell \in \Sigma$ be an unused letter
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**while** $U \notin \Sigma$ and $V \notin \Sigma$ **do**
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The problem with occurrences over the cut disappeared.
But equation changes, we need to bound its size.

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We show that

- we stay in $\mathcal{O}(n^2)$ space.
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- If there is something non-crossing: compress it.
- If there are only crossing: choose one that minimises the equation.

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**Lemma**

*There is always some choice to be $\leq 8n^2$.*

There are $m \leq 8n^2$ letters and $k \leq 2n$ different crossing blocks/pairs.
Some covers $\geq m/k$ letters.
Its compression removes $\geq m/2k$ letters and introduces $2n$ letters.
We are left with at most

$$(1 - 1/2k) \cdot m + 2n \leq (1 - 1/4n) \cdot 8n^2 + 2n = 8n^2 \quad .$$

# So what?

## Valid concern:

- We still have PSPACE, known for 20 years.
- Simplicity lies in the eye of the author
  (earlier version was more complex)

- I think it is simpler.
- No word combinatorics: much more robust.
  - easier to generalize
  - easier to adapt

# Representation of all solutions

## Plandowski

Graph-like representation

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## Recompression

Rather easy, various possible approach:

- Change the notion of minimality (like Plandowski): natural
- tricks with the alphabet (Diekert)

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Rather easy, various possible approach:

- Change the notion of minimality (like Plandowski): natural
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## EDT0L

The set of all solutions is an EDT0L language.
Requires algebraic reformulation, follows naturally.

# Free groups

Involution + regular constraints $\Longrightarrow$ equations in free group

## Free groups

- PSPACE algorithm (known)
- exponential representation of all solutions (new)
- set of all solutions is an EDT0L language

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Keep transition matrix for each letter.

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When we replace $ab$ we replace $\overline{ab}$ as well.
Some technicalities with self-involuting pairs.

## Regular constraints

Keep transition matrix for each letter.Some problems with the

length-minimal solutions
(we cannot erase letters)

# Required more work

- partial commutation [Diekert, J., Kufleitner]
- twisted word equations (permutation of letters) [Diekert, Elder]
- nondeterministic linear space = context sensitive language [J.]
- context unification (term equations) [J.]
- linear time for one variable [J.]

# Efficient variants

- equality of SLPs
- smallest grammar problem
- smallest tree grammar problem
- compressed membership for NFAs
- compressed pattern matching

# Open questions

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- Are word equations in NP?
- are smallest solutions at most doubly exponential?