k-Spectra of weakly-c-balanced Words

k-Spectra

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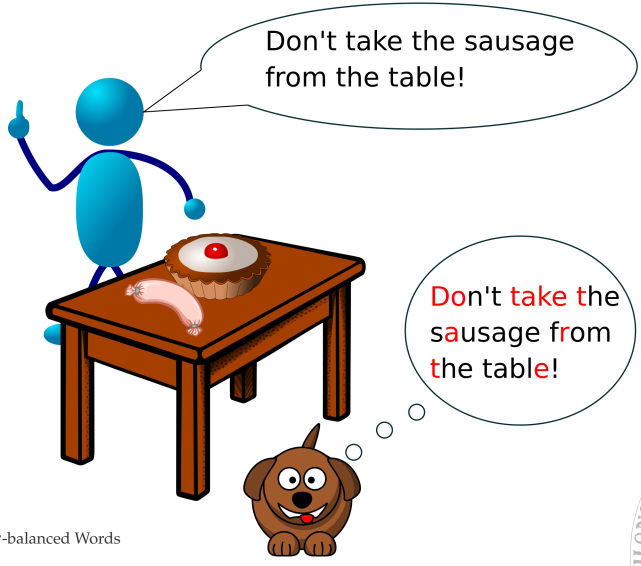
Information Loss



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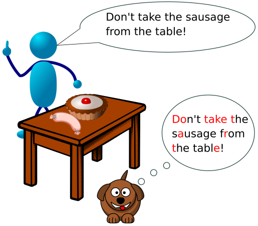

informal definition:
deleting arbitrary letters from a word (preserving the order) results in a scattered factor of this word

Scattered Factors

Definition (Scattered Factor, (Scattered) Subword)

$v = v_1 \dots v_n \in \Sigma^*$ scattered factor of w iff

$$\exists u_0 \dots u_n \in \Sigma^* : \ w = u_0 v_1 u_1 v_2 \dots v_{n-1} u_{n-1} v_n u_n .$$

k-Spectra

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Example: abba

$\{\mathsf{abba}\}$	4-spectrum
$\{\mathsf{aba}, \mathsf{bba}, \mathsf{abb}\}$	3-spectrum
$\{\mathsf{aa}, \mathsf{ab}, \mathsf{bb}, \mathsf{ba}\}$	2-spectrum
$\{\mathsf{a}, \mathsf{b}\}$	1-spectrum

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{aba, bba, abb}	3-spectrum
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{a, b}	1-spectrum

We are not considering multisets.

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Given $S \subseteq \Sigma^*$ decide whether S is the spectrum of some word w.

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Given a k-spectrum decide whether it is independent, *e.g.* $\{\mathtt{ab}, \mathtt{ba}, \mathtt{aa}\}$ *is not independent since* aa *can be deduced from* ab *and* ba.

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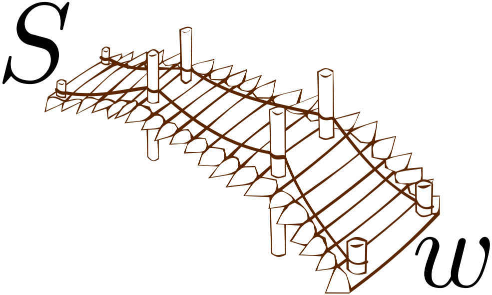
Problem

Determine the index of the equivalence relation that relates word with the same spectrum.

Middle Step Between S and w



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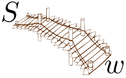

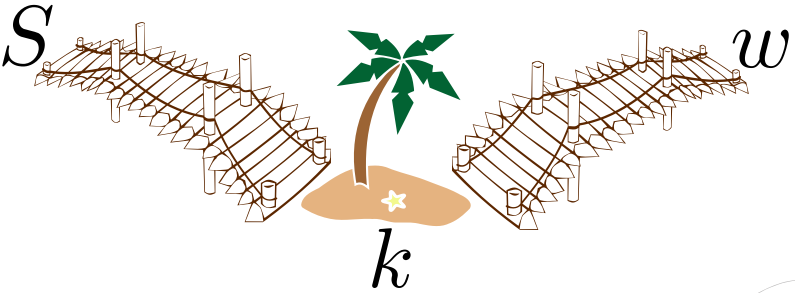

Reformulated Problem

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Decide for a given $n \in \mathbb{N}$ whether there exists $w \in \Sigma^*$ and $k \in \mathbb{N}$ with $|\mathrm{ScatFact}_k(w)| = n$.

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To start with we only consider a binary alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.

Examples

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- $n = 2^k, k \in \mathbb{N}$: $w = (\mathsf{ab})^k$
- n square number at least 4, $k := 2(\sqrt{n} - 1)$: $w = \mathsf{a}^{\frac{k}{2}}\mathsf{b}^k\mathsf{a}^{\frac{k}{2}}$

Weakly c-balanced words

Definition

Binary word $w \in \{\mathsf{a}, \mathsf{b}\}^*$ weakly c-balanced for a $c \in \mathbb{N}_0$ iff

$$\big|\, |w|_{\mathsf{a}} - |w|_{\mathsf{b}} \,\big| = c.$$

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Obviously for every $w \in \{\mathtt{a}, \mathtt{b}\}$ exists $c \in \mathbb{N}_0$ such that w is c-balanced.

Pecularities of Restriction to Cardinalities

- $\mathrm{ScatFact}_k(\overline{w}) = \{\overline{u} \mid u \in \mathrm{ScatFact}_k(w)\}$
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$\mathtt{a} < \mathtt{b}$ assumed: only consider the lexicographically smallest element in such an equivalence class

Solving the first problem

Theorem

For all $n \in \mathbb{N}$ the k-spectrum of $w = \mathtt{a}^k\mathtt{b}^k$ for $k = n-1$ has n elements, i.e. $|\mathrm{ScatFact}_{n-1}(\mathtt{a}^{n-1}\mathtt{b}^{n-1})| = n$.

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Proof:

- all $\mathtt{a}^r\mathtt{b}^s$ for $r+s = n-1$ are the scattered factors of length $n-1$

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- n possibilities $\square$

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For all $n \in \mathbb{N}$ *the* k*-spectrum of* $w = \mathsf{a}^k\mathsf{b}^k$ *for* $k = n-1$ *has* n *elements, i.e.* $|\mathrm{ScatFact}_{n-1}(\mathsf{a}^{n-1}\mathsf{b}^{n-1})| = n$.

Corollary

$S_n = \{\mathsf{a}^r\mathsf{b}^s \mid r+s = n \in \mathbb{N}\}$ *is a scattered factor set for all* $n \in \mathbb{N}$.

Partly Solving the Second Problem

Theorem

Given $k, n \in \mathbb{N}$ with $n - 1 \leq k$ set $c = k - n + 1$ and consider $w = \mathsf{a}^k\mathsf{b}^{k-c}$. Then for all $i \in [c]_0$ the $(k - i)$-spectrum of w has cardinality n.

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Proof:

- $i = 0$: $\mathsf{a}^r \mathsf{b}^s$ with $r + s = k \rightsquigarrow k - c + 1 = n$ possibilities
- $i \neq 0$: all the scattered factor are just *shortened* for the $(k - i)$-spectra

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- Given $n \in \mathbb{N}$ for each c we have $c+1$ different sets being a spectrum of cardinality n.

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We were not happy! We would like to fully characterise for given c and word-length which cardinalities are reachable.

Minimal Cardinality

Lemma

$w \in \Sigma^*, k, c \in \mathbb{N}_0$ with $c \leq k$:

$$\forall i \in [c]_0 : |\mathrm{ScatFact}_{k-i}(w)| = k - c + 1 \quad \textit{iff} \quad w = \mathtt{a}^k \mathtt{b}^{k-c}.$$

Moreover $|\mathrm{ScatFact}_{k-i}(w)| \geq k - c + 1$ for all $i \in [c]_0$

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Proof idea for remaining part:

- suppose $w \neq \mathsf{a}^k \mathsf{b}^{k-c}$ (neither one of the symmetric cases)
- $\Rightarrow w = w_1 \mathsf{aba} w_2$
- induction on word-length

Max. Card. for Weakly-0-Balanced Words of Length $2k$

Theorem

$w \in \Sigma^*$:

$$\mathrm{ScatFact}_k(w) = \Sigma^k \quad \textit{iff} \quad \{\mathsf{ab}, \mathsf{ba}\}^k \cap \mathrm{ScatFact}_{2k}(w) \neq \emptyset$$

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i.e. $w \in \{\mathsf{ab}, \mathsf{ba}\}^k$ iff $|\mathrm{ScatFact}_k(w)| = 2^k$

Max. Card. for Weakly-c-Balanced Words

unfortunately the result for the maximal cardinality is not generalisable in the same way as for the minimal one

k	ababa	abbaa	
1	Σ	Σ	prev. result
2	Σ^2	Σ^2	prev. result
3	$\Sigma^3 \setminus \{b^3\}$	$\{a^3, aba, ab^2, ba^2, b^2a\}$	

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for $c > 0$, the switches from a to b and v.v. matter!

Max. Card. for Weakly-c-Balanced Words

Theorem

For all $i \leq k - c$, $c \in [k]_0$, $k \in \mathbb{N}$

$$|\mathrm{ScatFact}_i((\mathsf{ab})^{k-c}\mathsf{a}^c)| = 1 + 2^{k-c} + \sum_{j \in [(i+c)-k-1]_0} |\mathrm{ScatFact}_{i-j-1}((\mathsf{ab})^{k-c-1}\mathsf{a})|$$

with $|\mathrm{ScatFact}_\ell(\mathrm{Pref}_n(\mathsf{ab})^\omega)| = \sum_{j \in [n-\ell]_0} \binom{\ell}{n-\ell-j}$

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Proof-Idea.

- $(i_1, \ldots, i_m)$ deleting sequence $\rightsquigarrow$ scattered factor
- several deleting sequences lead to the same scattered factor
- count only one of these sequences

k-spectra for weakly-0-balanced words of length $2k$

Properties of weakly-0-balanced words

- same amount of $\mathtt{a}$s and $\mathtt{b}$s

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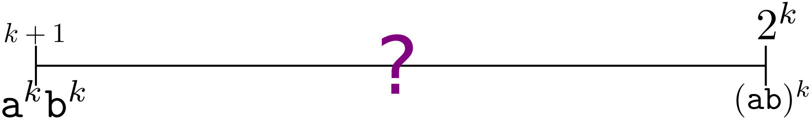
- the k-spectra has at most 2^k elements

Spectrum of k-spectra



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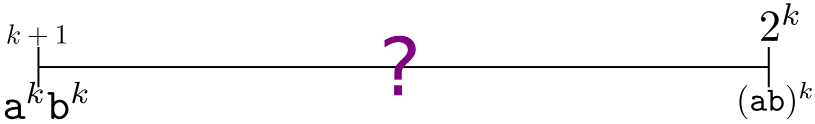

Proof for "$|\mathrm{ScatFact}_k(w)| = k + 1$ iff $w = \mathtt{a}^k\mathtt{b}^k$ gives also that $k + 2$ is not reachable!

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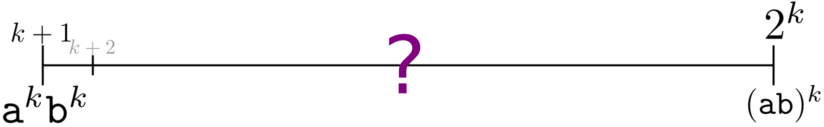

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Lemma

The k-spectrum of a weakly-0-balanced word $w \in \Sigma^*$ has cardinality $2k$ iff w is either $\mathtt{a}^{k-1}\mathtt{bab}^{k-1}$ or $\mathtt{a}^{k-1}\mathtt{b}^k\mathtt{a}$, i.e.

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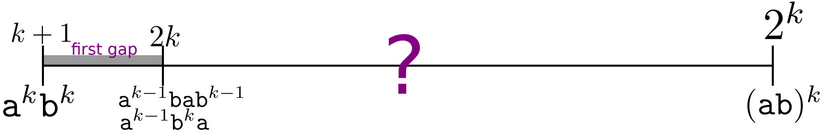
Our proof also shows

- If w is neither $\mathsf{a}^k\mathsf{b}^k$ nor $\mathsf{a}^{k-1}\mathsf{bab}^{k-1}$ nor $\mathsf{a}^{k-1}\mathsf{b}^k\mathsf{a}$, then the cardinality is greater than $2k$

Spectrum of k-spectra



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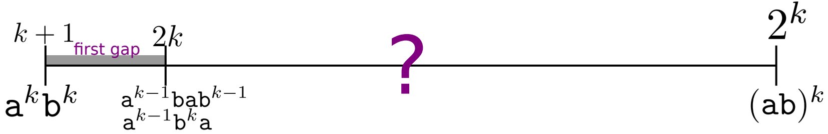
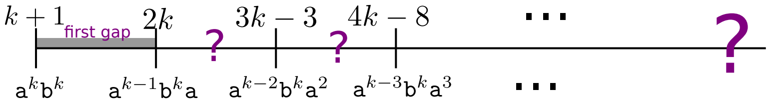


$\mathsf{a}^{k-1}\mathsf{b}^k\mathsf{a}$ generalisable to $\mathsf{a}^{k-i}\mathsf{b}^k\mathsf{a}^i$ for $i \in \left[\lfloor \frac{k}{2} \rfloor\right]$:

$$|\mathrm{ScatFact}_k(\mathsf{a}^{k-i}\mathsf{b}^k\mathsf{a}^i)| = k(i+1) - i^2 + 1$$

Spectrum of k-spectra



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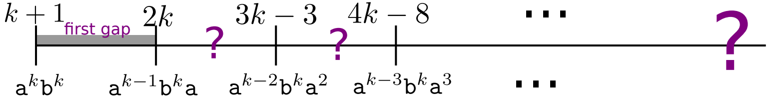
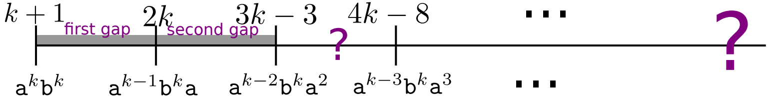


Promising news: the k-spectra of weakly-0-balanced words cannot have cardinality $2k + i$ for $i \in [k-4]$



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and this result is generalisable

The Thing in the "Gap"

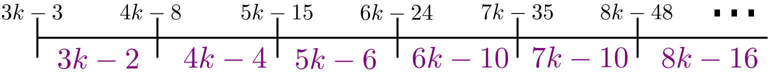
Lemma

For $k \geq 5$ *and* $i \in [k-1]$

- $|\mathrm{ScatFact}_k(\mathtt{a}^{k-2}\mathtt{b}^i\mathtt{a}\mathtt{b}^{k-i}\mathtt{a})| = k(2i+2) - 6i + 2$
- $|\mathrm{ScatFact}_k(\mathtt{a}^{k-2}\mathtt{b}^i\mathtt{a}^2\mathtt{b}^{k-i})| = k(2i+1) - 4i + 2$

Spectrum of k-spectra

$k \geq 38$



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Proof:

- "$\Leftarrow$" $\surd$
- "$\Rightarrow$" if there is a scattered factor not of the form $\mathsf{b}^{i+1}\mathsf{a}^{k-i-1}$ then less than $2^k - 1$ element are in the k-spectrum

Overview for weakly-0-balanced words

k=	2	3	4	5	6	7	8	9
	3	4	5	6	7	8	9	10
	4	5	6	7	8	9	10	11
		6	7	8	9	10	11	12
		7	8	9	10	11	12	13
		8	9	10	11	12	13	14
			10	11	12	13	14	15
			11	12	13	14	15	16
			12	13	14	15	16	17
			13	14	15	16	17	18
			14	15	16	17	18	19
			15	16	17	18	19	20
			16	17	18	19	20	21
				18	19	20	21	22
				19	20	21	22	23
				20	21	22	23	24
				21	22	23	24	25
				22	23	24	25	26
				23	24	25	26	27
				24	25	26	27	28

9	8	7	6	5	4	3	2	=k
512	256	128	64	32	16	8	4	
511	255	127	63	31	15	7	3	
510	254	126	62	30	14	6		
509	253	125	61	29	13	5		
508	252	124	60	28	12	4		
507	251	123	59	27				
506	250	122	58	26				
505	249	121	⋮	25				
504	248	⋮	38					
503	⋮	54	37					
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82	39							
81								