Relation Algebra: Decidability and Axiomatizations

The case of Kleene lattices

August 6th, 2019

Amina Doumane
Warsaw university

Binary relations are everywhere

- **Graph theory**
- **Semantics of imperative programs**
- **Foundations of mathematics**

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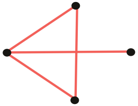

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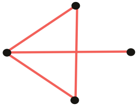

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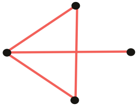

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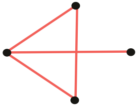

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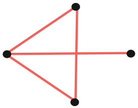

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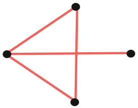

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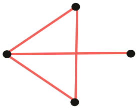

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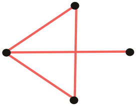

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$x \leftarrow 1; (y \leftarrow x) \oplus (y \leftarrow 0);$

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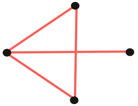

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- **Foundations of mathematics**

Two binary relations	$\in$ (membership), 1 (identity)
Operations	$\cup$ (union), $\cdot$ (composition), $\smile$ (converse), ^{c} (complement)
Sentences	$e = f$

Relation algebra

Relational Operators

identity relation	:	1
empty relation	:	0
composition	:	$R \cdot S$
union	:	$R \cup S$
intersection	:	$R \cap S$
trans. closure	:	R^+
converse	:	$R^\smile$
complement	:	R^c

Relation algebra and their universal laws

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$$\mathrm{Rel} \models R \cdot (S \cdot R)^{+} \;=\; (R \cdot S)^{+} \cdot R$$

$$\mathrm{Rel} \models 1 \;\cup\; R^{\star} \cdot S \;\subseteq\; (R \cup S)^{\star}$$

$$\mathrm{Rel} \models (R \cap S) \cdot T \;\subseteq\; (R \cdot T) \cap (S \cdot T)$$

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Decidability and **Axiomatizability**

Deciding the equational theory of Relation Algebra

Decidability problem

Input: Expressions e and f.
Output: Is $\mathrm{Rel} \models e = f$ a universal law?

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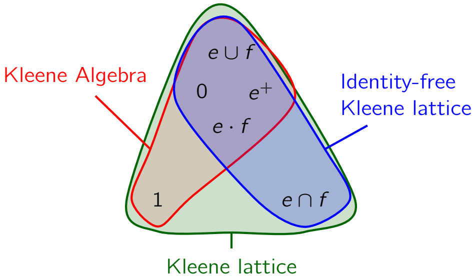

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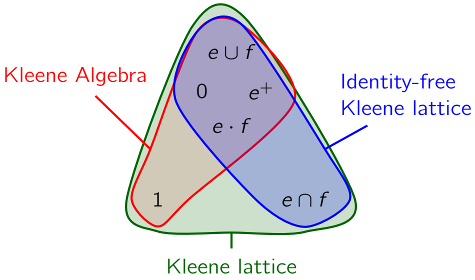

KA	PSPACE-complete	(Kleene & Pratt 80 and Meyer & Stockmeyer 72)
KL^-	EXPSPACE-complete	(Brunet & Pous 15)
KL	EXPSPACE-complete	(Nakamura 16)

Axiomatizing the equational theory of Relation Algebra

Axiomatization

- **A set of axioms of the form**

$$e = f \qquad \text{or} \qquad e = f \Rightarrow g = h$$

- **Deduction rules**

$$e = f \ \wedge \ f = g \ \Rightarrow \ e = g \qquad \text{and} \qquad e = f \Rightarrow e\sigma = f\sigma$$

Axiomatizing the equational theory of Relation Algebra

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Find a set of (quasi-)equations axiomatizing the equational theory of relations.

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Why this is important?

- Extend decision procedure from relation algebras to other algebras
- Solve hard instances by hand
- Gives certificates

Axiomatizing the equational theory of Relation Algebra

Axiomatization problem

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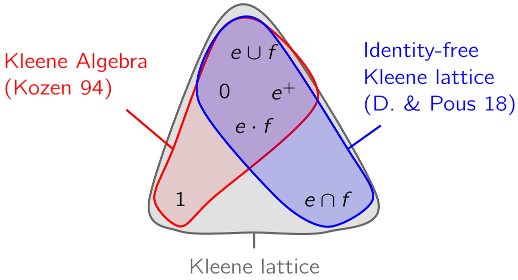

KA expressions & languages

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

KA expressions

$$e, f \in ::= 0 \mid 1 \mid a \mid e \cdot f \mid e \cup f \mid e^+$$

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Language of a regular expression $\mathcal{L}(e)$

$$\mathcal{L}(a \cdot (b \cup c)) = \{ab, ac\} \qquad \mathcal{L}(a \cdot 0) = \emptyset$$

$$\mathcal{L}\left(a^+\right) = \{a, aa, \dots\}$$

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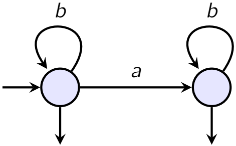
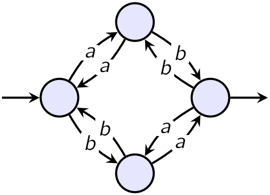
Theorem (Pratt 1980)

$$\mathrm{Rel} \models e \subseteq f \;\Leftrightarrow\; \mathcal{L}(e) \subseteq \mathcal{L}(f)$$

Non-deterministic Finite Automata (NFA)

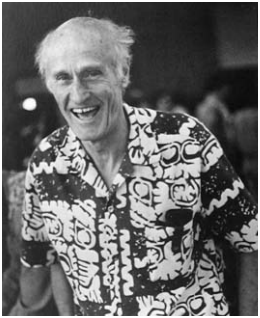




Non-deterministic Finite Automata (NFA)

Kleene Theorem

A language is regular if and only if it is recognized by an automaton.



Stephen Cole Kleene

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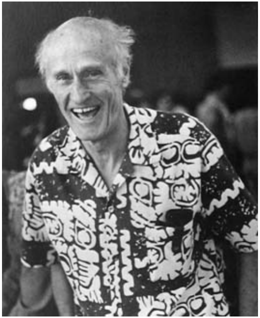

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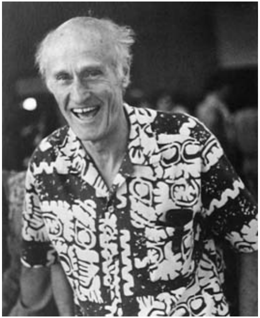
Corollary

Relational equivalence is decidable for regular expressions.



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Axiomatization

Axioms of Kleene Algebra

- Axioms of an idempotent semiring decribing the behaviour of $\cup, \cdot, 0, 1$.
- Two axioms describing the behaviour of $^+$:

$$f \cdot e \cup f \subseteq f \quad \Rightarrow \quad f \cdot e^+ \cup f \subseteq f$$
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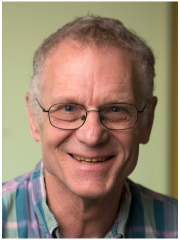
We write $\mathsf{KA} \vdash e \subseteq f$
if $e \subseteq f$ **follows from the axioms of Kleene Algebra.**



Dexter Kozen

Theorem (Kozen 1994)

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad KA \vdash e \subseteq f$$

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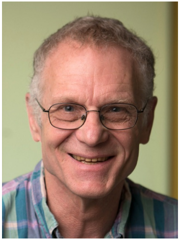

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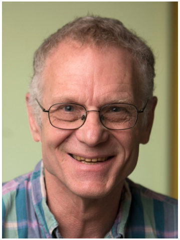

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Theorem (Completeness)

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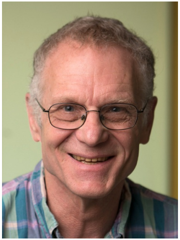

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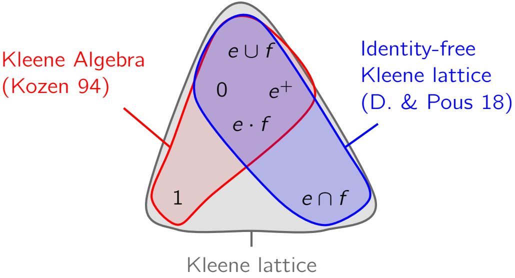
Theorem (Completeness)

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad \mathcal{L}(e) \subseteq \mathcal{L}(f) \quad \Rightarrow \quad KA \vdash e \subseteq f$$


Kleene Algebra
(Kozen 94)
Identity-free
Kleene lattice
(D. & Pous 18)
$e \cup f$
0
e^+
$e \cdot f$
1
$e \cap f$
Kleene lattice

KL^- expressions & languages

Let $\Sigma = \{a, b, \dots\}$ be a finite alphabet.

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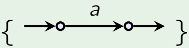
Another notion of language is needed!

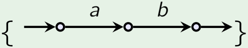
Language of a KL^{-} expression

Graph language of an expression $\mathcal{G}(e)$

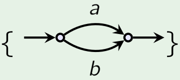
$\mathcal{G}(a) =$



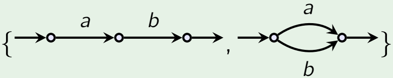
$\mathcal{G}(a \cdot b) =$



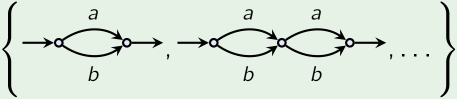
$\mathcal{G}(a \cap b) =$



$\mathcal{G}(a.b \cup a \cap b) =$



$\mathcal{G}\left((a \cap b)^{+}\right) =$



Characterization theorem

$$\mathrm{Rel} \models e \subseteq f \overset{?}{\Leftrightarrow} \mathcal{G}(e) \subseteq \mathcal{G}(f)$$

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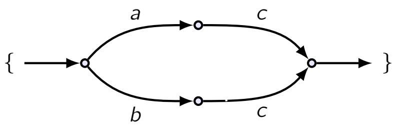
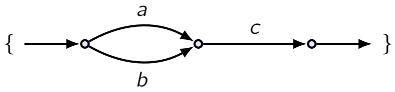
$$\mathrm{Rel} \models (a \cap b) \cdot c \quad \subseteq \quad (a \cdot c) \cap (b \cdot c)$$



$(a \cdot c) \cap (b \cdot c)$



$(a \cap b) \cdot c$

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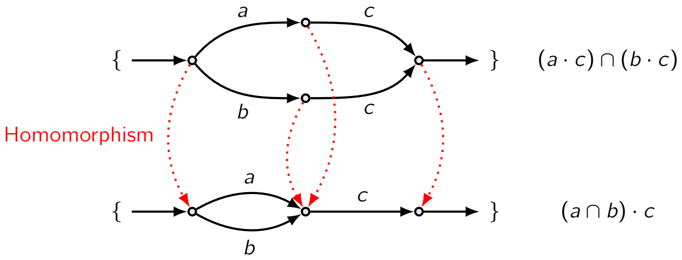

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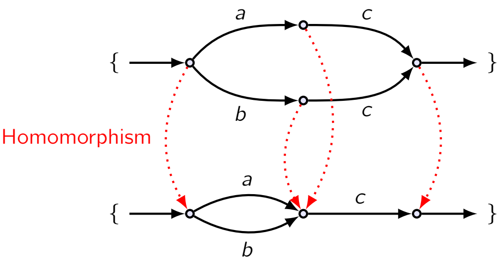

$(a \cdot c) \cap (b \cdot c)$

$(a \cap b) \cdot c$

Paul Brunet

Damien Pous

Theorem [Brunet & Pous, LICS 2015]

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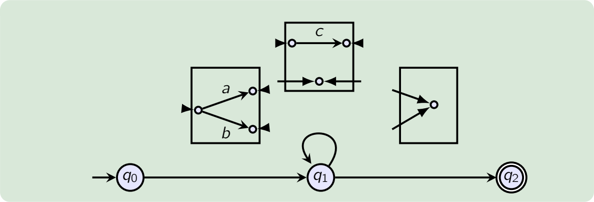
Petri automata



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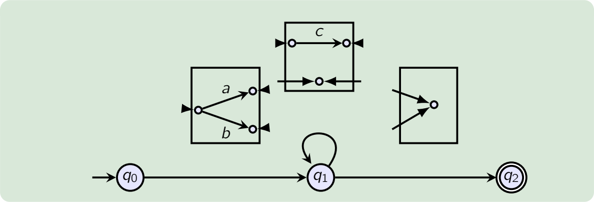
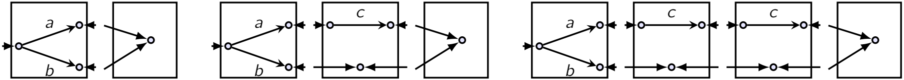
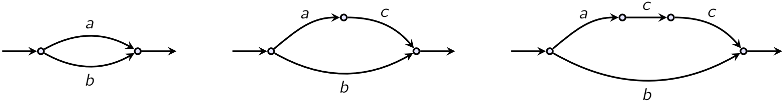


Runs:





Kleene theorem

Theorem (Brunet & Pous LICS 2015)

For every Petri automaton P, there is an expression e such that $\mathcal{G}(e) = \mathcal{G}(P)$.

Proof.

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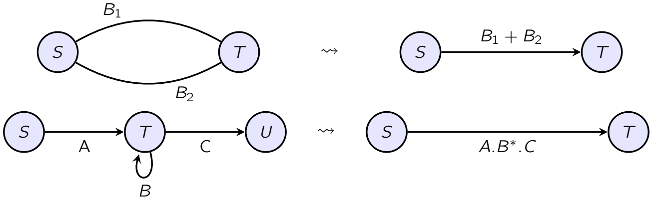

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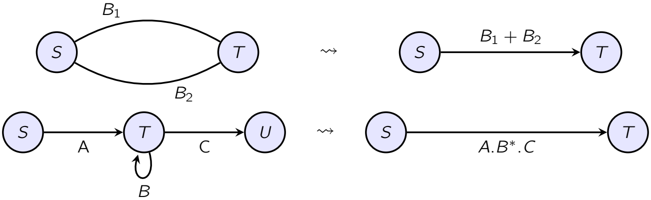
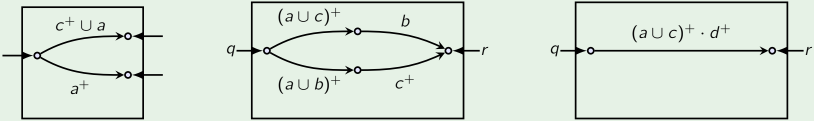


Generalized slices of graphs



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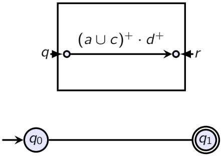

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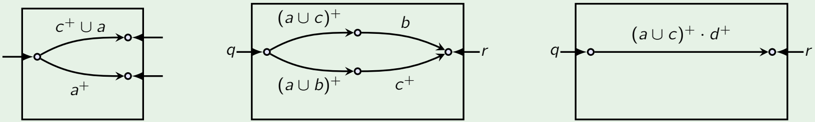

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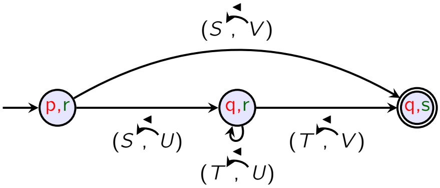

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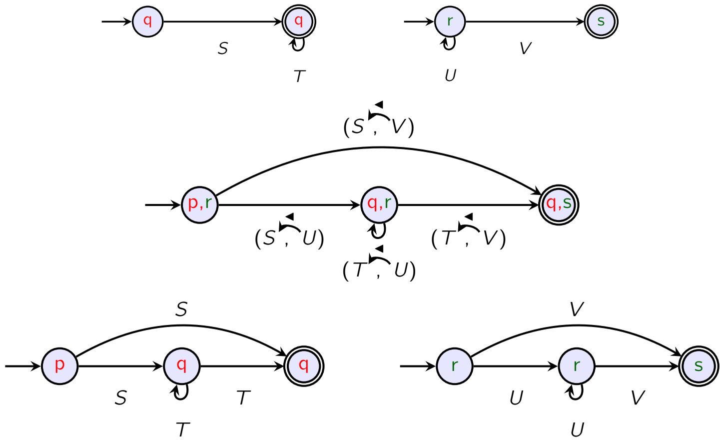

Axiomatization

Axioms of identity-free Kleene lattices

- Axioms of Kleene algebra.
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Theorem (D. & Pous 2018)

$$\mathrm{Rel} \models e \subseteq f \quad \Leftrightarrow \quad KL^- \vdash e \subseteq f$$

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A strengthening of KA completeness.

Synchronised Kleene theorem

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If P and Q are Petri automata such that $\mathcal{G}(P) \overset{\blacktriangleleft}{\subseteq} \mathcal{G}(Q)$, there are two expressions e and f such that:

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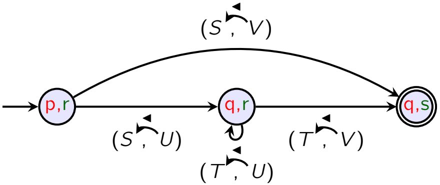

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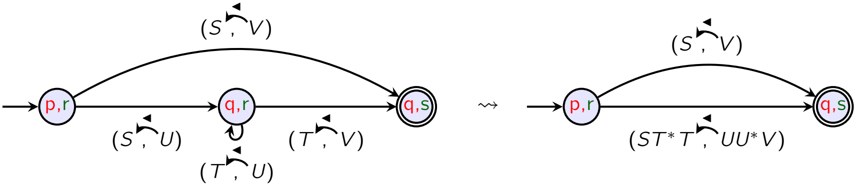

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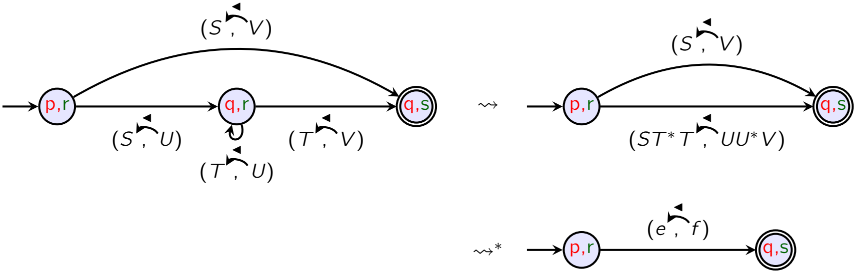
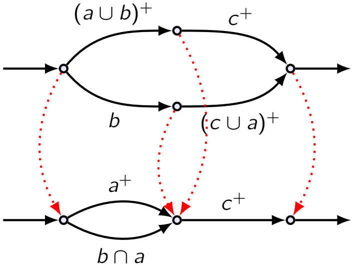


Generalized graph homomorphism



$KL^- \vdash a^+ \subseteq (a \cup b)^+$

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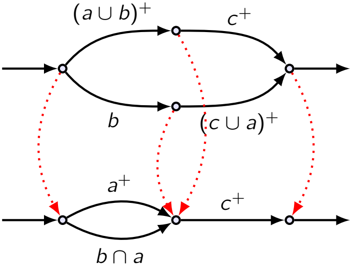

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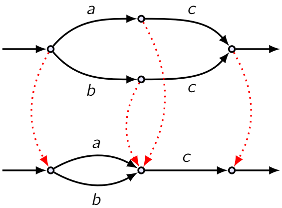

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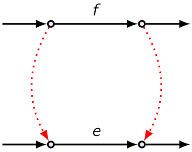

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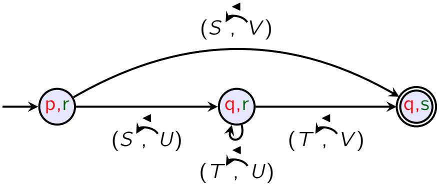

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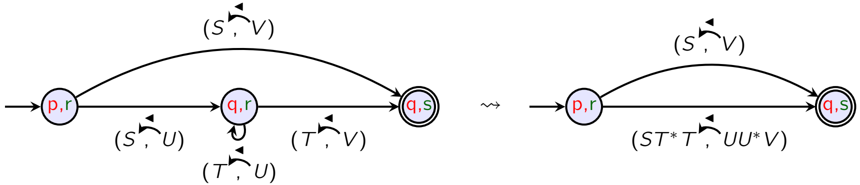

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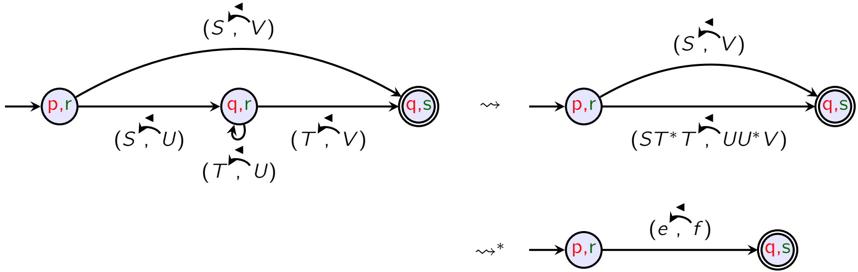

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e $\qquad$ $\mathrm{Rel} \models e \subseteq f$ $\qquad$ f

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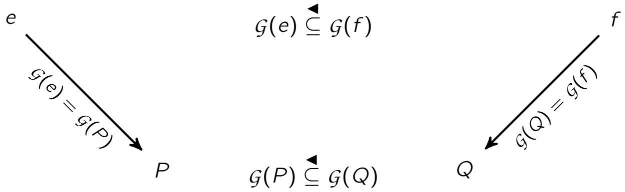

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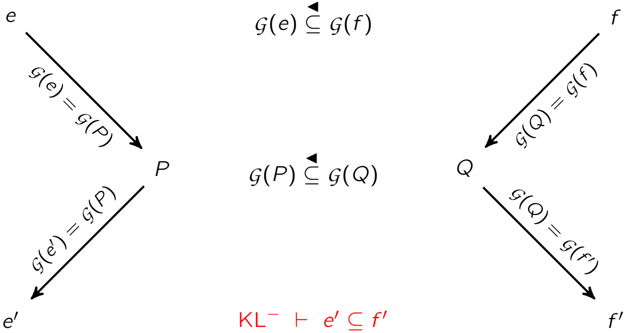

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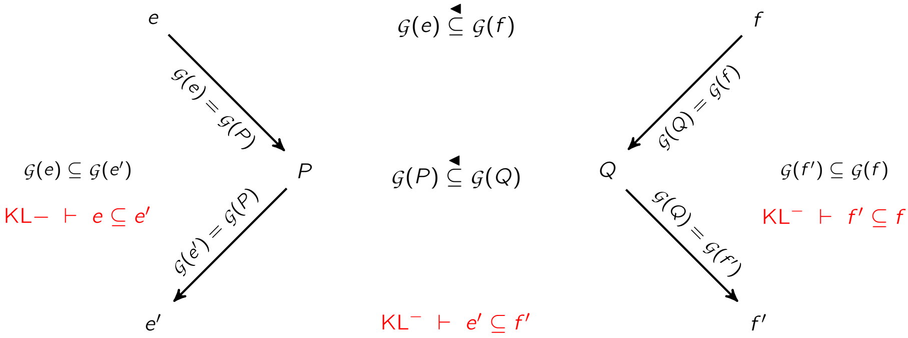
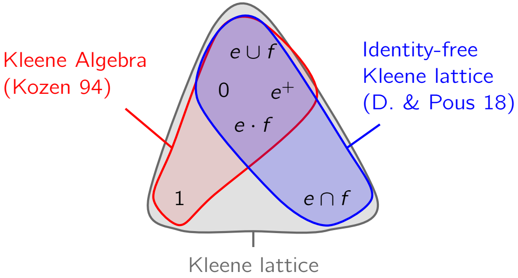


Conclusion & Future work

- Axiomatizing positive relation algebras



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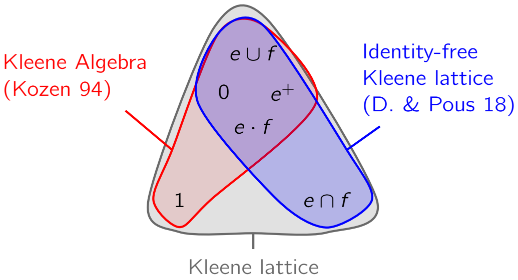

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Thank you for your attention !