# A linear bound on the k-rendezvous time for primitive sets of NZ matrices

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# Outline

**Intro.**

- Synchronizing automata, Černý conjecture, the $k$-rendezvous time
- Primitive sets of matrices (aka primitive nondeterministic finite state automata), the $k$-rendezvous time

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- Upper bound on the $k$-rendezvous time of NZ primitive nondeterministic finite state automata
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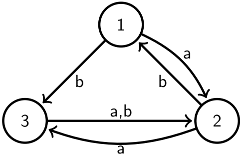
**Conclusion.**

The **k-rendezvous time** of an NZ primitive nondeterministic finite state automaton on $n$ states is **at most linear in n** for any fixed and small enough $k$.

# Synchronizing automata




**Deterministic Finite Automaton (DFA):**

$\mathcal{A} = \langle Q, \Sigma, \delta \rangle$, $\delta : Q \times \Sigma \to Q$, $\delta(q, a) = q.a$.

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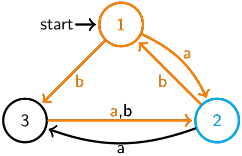


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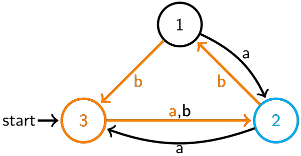


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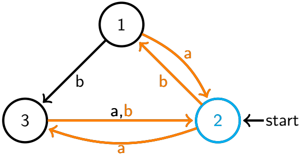


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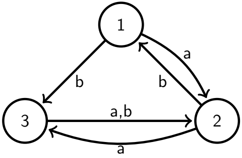


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**Synchronizing word:**

$w = abba$ , $\forall q \in Q$, $q.w = 2$

A DFA is **synchronizing** if there exists a word $w$ and a state $t \in Q$ such that for any state $q$, $q.w = t$.

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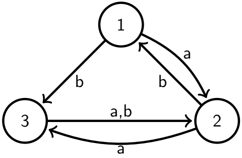


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**Equivalently:**

A DFA is a finite set of row-stochastic $\{0, 1\}$-matrices and it is **synchronizing** iff there exists a product of its matrices that has an **all-ones column**.

# The Černý conjecture

**The Černý conjecture (1964):** any synchronizing DFA on $n$ states has a synchronizing word of length $\leq (n-1)^2$.

- Confirmed by exhaustive search for small $n$ [Trahtman '16, Bondt et. al. '17].
- Proved for some classes of automata [Kari '03, Carpi '08, ...].
- Best upper bound is cubic in $n$ [Pin-Frankl '83, Szykuła '18].

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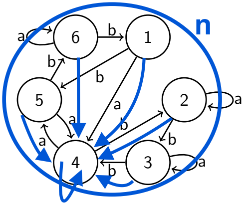


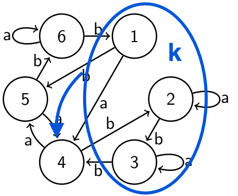

# The k-rendezvous time

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**Relaxation of the problem: upper bounds on the k-RT?**

- $k = 2$ trivial. 1-RT=1
- $k = 3$, quadratic upper bound in $n$ [Gonze, Jungers '16]
- $k \geq 4$ ?
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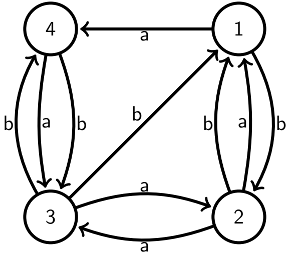
**Our approach:**

We extend the notion of k-RT to **primitive nondeterministic finite state automata (NFAs)**.

# Primitive NFAs

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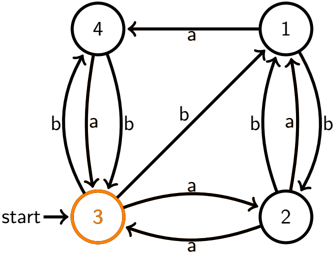


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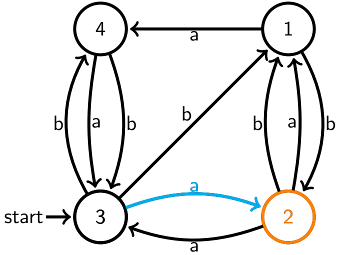


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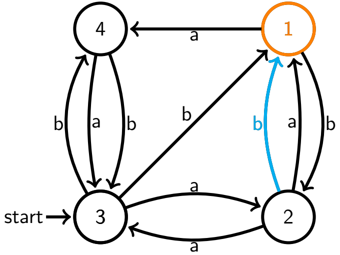


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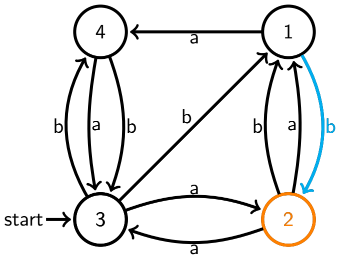


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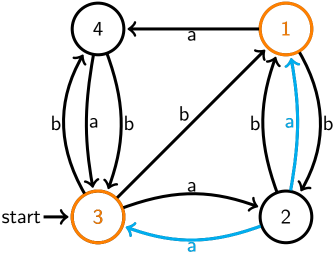


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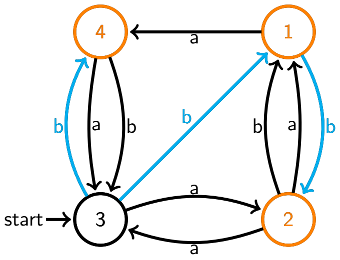


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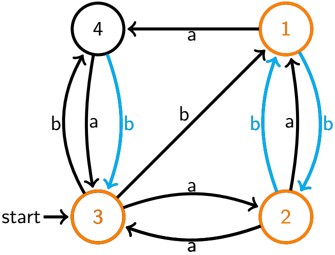


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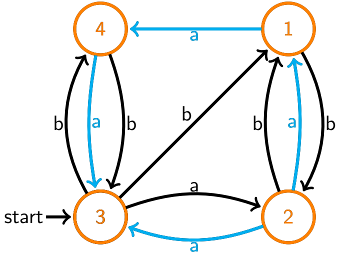


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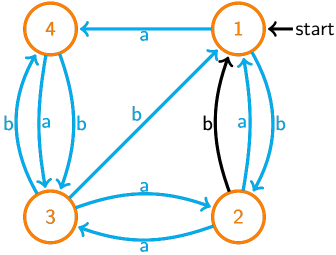


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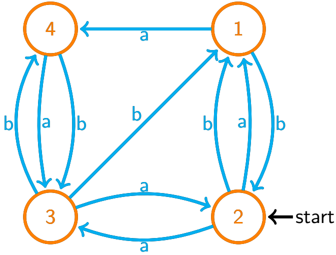


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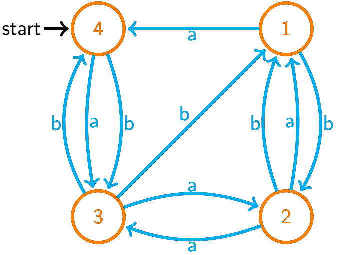


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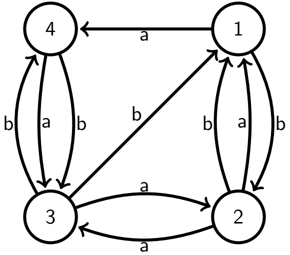


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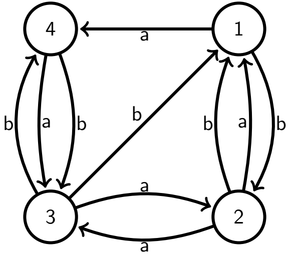


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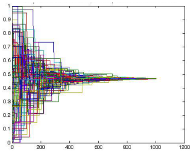
An NFA is a finite set of $\{0, 1\}$-matrices and it is **primitive** iff there exists a product of its matrices that is **entrywise positive**.

# Applications of primitivity

**Primitivity as a property of set of matrices finds application in**:

1. Consensus for discrete-time multi-agent systems [Chevalier et. al. '15];
2. Stochastic switched system [Jungers, Protasov '12];
3. Time-inhomogeneous Markov chains [Seneta '81, Costa et. al. '05];
4. **Synchronizing automata** [Blondel et. al. '15, Gerencsér et. al. '16, C., Jungers '18];



$$x(t+1) = A_{\sigma(t)}x(t)$$
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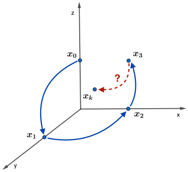
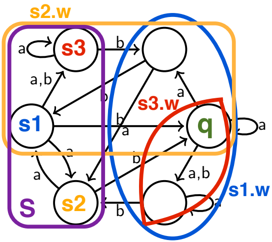


# Relaxation of the problem

**The k-RT for primitive NFAs:** length of the shortest matrix product having a row or a column with at least $k$ positive entries.

**In automata terms:** length of the shortest word $w$ for which there exists $q \in Q$ and $S \subset Q$, $|S| = k$, such that one of the following holds:

**1.** for all $s \in S$, $\mathbf{q} \in \mathbf{s.w}$;

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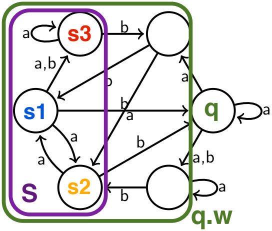

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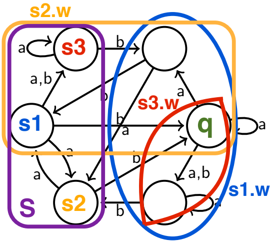


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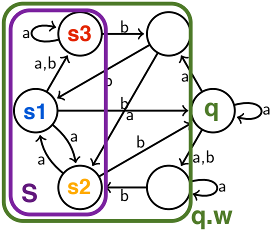


**In this talk:** we present an upper bound on the k-RT of NZ primitive NFAs that depends on $k$ and the number of states $n$

# Motivations: linking results

An NFA is **NZ** if $\forall a \in \Sigma$ and $\forall q \in Q$, $q$ has both an in-going and an out-going edge labeled by $a$.
**Equiv.:** if every matrix has a positive entry in every row and column.

$$\mathcal{M} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

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$$\swarrow$$

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$$\mathcal{A}_{\mathcal{M}} = \left\{ \overbrace{\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}}, \overbrace{\begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}} \right\}$$

$$\mathcal{M}^T = \left\{ \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&1&0&1\\ 1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 1&0&0&0\\ 0&0&0&1\\ 1&0&1&0 \end{pmatrix} \right\}$$

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# Motivations: linking results

**Notation:** $\mathcal{A}$ a synchronizing DFA and $\mathcal{M}$ a primitive NFA.

* $\mathbf{rt_k}(\mathcal{A})$: length of the shortest word of $\mathcal{A}$ mapping $k$ states onto one
* $\mathbf{rt_k}(\mathcal{M})$: length of the shortest word $w$ of $\mathcal{M}$ having a row or a column with $k$ positive entries

**Proposition.** Given an *NZ* primitive NFA on $n$ states $\mathcal{M}$, the two DFAs $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{A}_{\mathcal{M}^T}$ are synchronizing and:

$$rt_k(\mathcal{M}) = \min\{rt_k(\mathcal{A}_{\mathcal{M}}), rt_k(\mathcal{A}_{\mathcal{M}^T})\}.$$

# Main result: upper bound on the $k$-RT

**Theorem.** Let $\mathcal{M}$ be an $NZ$ primitive NFA on $n$ states. Then for any $2 \le k \le n$ it holds that

$$rt_k(\mathcal{M}) \le B_k(n)$$

where

$$B_k(n) = \begin{cases} \dfrac{n(k^3 - 3k^2 + 8k - 12)}{6} + 1 & \text{if } 2 \le k \le \lfloor \sqrt{n} \rfloor \\ B_{\lfloor \sqrt{n} \rfloor}(n) + \dfrac{n(n+2)(k - \lfloor \sqrt{n} \rfloor)}{2} - \dfrac{n^2}{2} \displaystyle\sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} \frac{1}{i} & \text{if } \lfloor \sqrt{n} \rfloor + 1 \le k \le \lfloor \frac{n}{2} \rfloor \\ B_{\lfloor \frac{n}{2} \rfloor}(n) + \dfrac{(k - \lfloor \frac{n}{2} \rfloor) n^2}{2} & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \le k \le n \end{cases}.$$

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$rt_k(\mathcal{M})$ grows at most linearly in $n$ for any fixed $k \leq \sqrt{n}$.

# Intermediate result (I)

- **weight** of a nonnegative vector: number of its positive entries
- **supp(v)**= indices of the positive entries of the nonnegative vector $v$

**Proposition 1.** Let $\mathcal{M}$ be an *NZ* primitive NFA on $n$ states. Then for any $2 \le k \le n-1$ it holds that $rt_k(\mathcal{M}) \le B_k(n)$ where

$$\begin{cases} B_2(n) = 1 \\ B_{k+1}(n) = B_k(n) + n(1+n-a_k^n)/2 \quad \text{for } 2 \le k \le n-1. \end{cases}$$

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- $a_k^n = \min_{A \in S_n^k} a_k^n(A)$;
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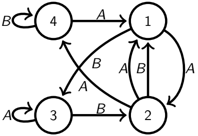
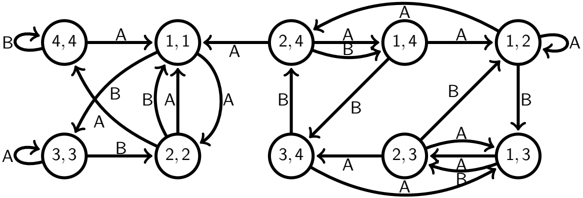
# Intermediate result (I): sketch of the proof

$$\mathcal{M} = \left\{ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$




**Square Graph (SG) for NFAs:** $M \in \mathcal{M}$

$(i,j) \xrightarrow{M} (i',j')$ iff $M_{i,i'} > 0$ and $M_{j,j'} > 0$, or $M_{i,j'} > 0$ and $M_{j,i'} > 0$.



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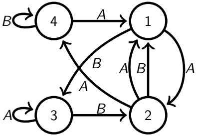


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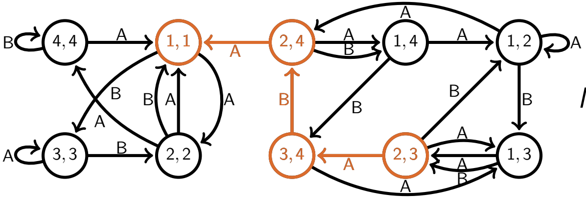


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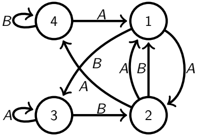


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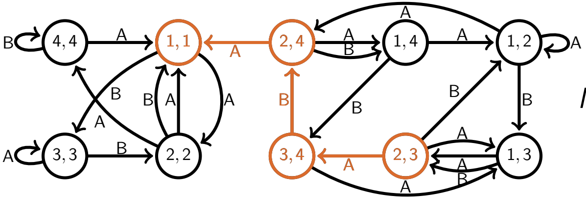


for any $NZ$ matrix $M$, $MABA_{*1} \geq M_{*2} + M_{*3}$

**In general:** a path $(i,j) \to (k,k)$ labeled by $A_{l_1} \ldots A_{l_s}$ means that $MA_{l_1} \cdots A_{l_s}$ has the $i,j$-th columns of $M$ summed up in column $k$. $\Rightarrow$ we will use this to construct a matrix with a column of weight $\geq k+1$ starting from a matrix with a column of weight $k$.

# Intermediate result (I): sketch of the proof

$\mathcal{M}^d=$ products of matrices of $\mathcal{M}$ of length $\leq d$.

**By induction on k**.

- **k** = **2**: $rt_2(\mathcal{M}) = 1 = B_2(n)$ trivial.

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$\ast$ Each $W^i$ has at least $a_k^n$ columns $c_i^1, \ldots, c_i^{a_k^n}$ whose support $\notin supp(W^i_{*i}) \Rightarrow$ by summing them up we get a column of weight $\geq k+1$ !

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$\ast$ We need to estimate the length of the shortest path in SG from a vertex in $\{(i, c_i^j)\}_{\substack{1\leq i\leq n \\ 1\leq j\leq a_k^n}}$ to a vertex of type $(k,k) \Rightarrow$ it is at most $n(1-n-a_k^n)/2$

□

# We have proven that

**Proposition 1.** Let $\mathcal{M}$ be an *NZ* primitive NFA on $n$ states. Then for any $2 \leq k \leq n-1$ it holds that $rt_k(\mathcal{M}) \leq B_k(n)$ where

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The proposition also holds if we **replace** $a_k^n$ **by any function** $b(n,k)$ such that $\mathbf{1 \leq b(n,k) \leq a_k^n}$.

# Intermediate result (II)

**Proposition 2.** For any $n$, $k$ integers s.t. $n \geq 3$ and $2 \leq k \leq n-1$,

$$a_k^n \geq \max\{n - k(k-1) - 1, \lceil (n-k)/k \rceil, 1\}.$$

Therefore it holds that $rt_k(\mathcal{M}) \leq B_k(n)$ where $B_k(n)$ satisfies the following recursion:

$$B_{k+1}(n) = \begin{cases} 1 & \text{if } k = 1 \\ B_k(n) + n(1 + k(k-1)/2) & \text{if } 2 \leq k \leq \lfloor \sqrt{n} \rfloor \\ B_k(n) + n(1 + n(k-1)/2k) & \text{if } \lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor n/2 \rfloor \\ B_k(n) + n^2/2 & \text{if } \lfloor n/2 \rfloor + 1 \leq k \leq n-1 \end{cases}.$$

By solving the above recurrence, we obtain the main result i.e. the upper bound on the k-rendezvous time for *NZ* primitive NFAs.

# Our bound on $a_k^n$ is optimal

**Proposition.** For any $n$, $k$ integers s.t. $n \geq 3$ and $2 \leq k \leq n-1$, it holds that

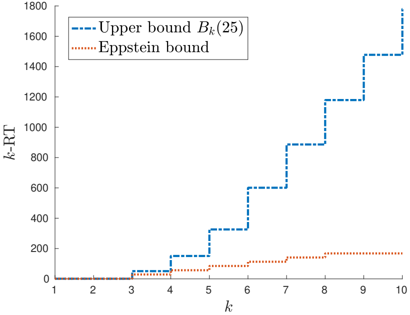
$$a_k^n = \max\{n - k(k-1) - 1, \lceil (n-k)/k \rceil, 1\}.$$

We **cannot** improve the bound $B_k(n)$ on the k-RT by improving the estimate on $a_k^n$, so new strategies are needed.

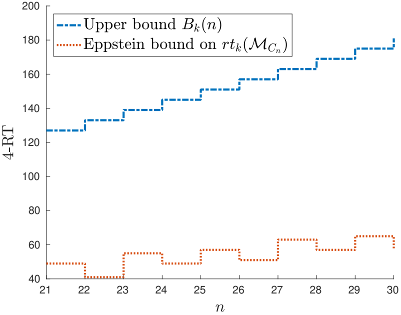
# Numerical experiments

* $\mathcal{M}_{\mathbf{C_n}}$: family of *NZ* primitive NFAs on *n* states with quadratic shortest primitive word [C., Jungers '18].
* **Eppstein bound**: greedy algorithm for approximating the k-RT of an *NZ* primitive NFA.

$n = 25$




$k = 4$



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- Our result is also an upper bound on $\min\{rt_k(\mathcal{A}_{\mathcal{M}}), rt_k(\mathcal{A}_{\mathcal{M}^T})\}$;
- Our technique cannot be improved as it already takes into account the **worst cases**, so new strategies are needed.

*...Thank you!*

*Questions?*